# Discrete Time Reward Processes, Stochastic Annuities and Insurance Models

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**Abstract.** The Markov and semi-Markov reward processes are a very powerful tool. They can be applied in many different fields, like mechanical systems, evaluation of computer systems etc. But in authors' opinion the most fruitful and natural application environment of these tools is the insurance field. In the paper will be given the definition of stochastic annuity and of its generalization their strict relation to the homogeneous and non-homogeneous semi-Markov reward processes. At last it will be shown how is natural to apply these rewards processes in the insurance environment.
**Keywords:** semi-Markov processes, Markov processes, stochastic annuities.

## 1 Introduction

Homogeneous semi-Markov processes (HSMP) were defined in the fifties. At beginning their applications were in engineering field, mainly in problem of reliability and maintenance see for example [Howard, 1971]. Non-homogeneous semi-Markov processes were defined in [Iosifescu Manu, 1972]. Applications of the semi-Markov processes were presented in finance and insurance see for example in [Janssen, 1966], [CMIR12, 1991]. Some of these applications were done attaching a reward structure to the process. This structure can be thought as a random variable associated with the state occupancies and transitions [Howard, 1971]. Non-homogeneous semi Markov reward processes were defined in [De Dominicis *et al.*, 1986]. The non-homogeneity results of great relevance in actuarial field, because in this way it is possible to take into account the different behaviour in function of the age. A stochastic approach to the annuity was given in [Wolthuis, 2003]. In this book the continuous time non-homogeneous Markov processes were used to generalize the annuity concept. This approach did not use the reward environment. The non-homogeneous Markov model was used to solve the multiple state insurance problems see [Wolthuis, 2003]. The aim of this paper is to investigate the strict correlation existing between insurance models and reward processes in a Markov or semi-Markov environment. In this light the financial concepts of stochastic annuities and generalized stochastic annuities are given. It is shown how these concepts correspond from financial view point to the reward processes. For an applicative intent, the paper is in discrete time environment. The paper begins introducing the semi-Markov processes both in homogeneous and non-homogeneous case. In section 3 the Discrete Time Homogeneous and Non-Homogeneous Markov and Semi-Markov ReWard Processes are presented. The subsequent section defines the concepts of stochastic annuity and of generalized stochastic annuity. In this part it is also explained the strict connection between the reward processes presented before and the annuities. In the last section the relations between multiple states insurance models and stochastic annuity are given.

## 2 Discrete time homogeneous and non-homogeneous semi-Markov processes.

In this part will be shortly described both the DTHSMP and DTNHSMP.

Let $E = \{1, 2, \ldots, m\}$ be the set of states of our system, $J_n \in E$ the random variable (r.v.) representing the state at the $n$ th transition and $T_n \in \mathbb{N}$ an other r.v. with set of states equal to $\mathbb{N}$ where $T_n$ represents the time of the $n$ th transition. It results:

$$J_n : \Omega \to E \quad T_n : \Omega \to \mathbb{N}$$

The process $(J_n, T_n)$ is a homogeneous (non-homogeneous) markovian renewal process if the kernel $\mathbf{Q} = [Q_{ij}(t)](\mathbf{Q} = [Q_{ij}(s,t)]])$ associated to the process is defined in the following way:

$$Q_{ij}(t) = P[J_{n+1} = j, T_{n+1} - T_n \le t | J_n = i]$$
$$(Q_{ij}(s,t) = P[J_{n+1} = j, T_{n+1} \le t | J_n = i, T_n = s])$$

Furthermore it is necessary to introduce the probability that the process will leave the state $i$ in a time $t$:

$$H_i(t) = \sum_{j=1}^{m} Q_{ij}(t) \quad \left( H_i(s,t) = \sum_{j=1}^{m} Q_{ij}(s,t) \right)$$

Furthermore the probabilities that there is a transtion at time $t$ are considered:

$$b_{ij}(t) = \begin{cases} Q_{\mathrm{ij}}(t) = 0 \text{ if } t = 0 \\ Q_{ij}(t) - Q_{ij}(t-1) \text{ if } t > 0 \end{cases}$$

$$\left(b_{ij}(s,t) \;=\; \begin{cases} Q_{ij}(s,t)=0 \;\text{ if }\; t \le s \\ Q_{ij}(s,t)-Q_{ij}(s,t-1) \;\text{ if } t > s \end{cases}\right)$$

Now it is possible to define the probability distribution of the waiting time in each state $i$, given that the state successively occupied is known:

$$F_{ij}(t) = P[T_{n+1} - T_n \le t | J_n = i, J_{n+1} = j]$$
$$(F_{ij}(s,t) = P[T_{n+1} \le t | J_n = i, J_{n+1} = j, T_n = s]).$$

Now the DTHSMP (DTNHSMP) $Z = (Z_t, t \in \mathbb{N})$ can be defined. It represents, for each waiting time, the state occupied by the process. The transition probabilities are defined in the following way:

$$\phi_{ij}(t) = \mathrm{P}\left[Z_t = j | Z_0 = i\right]$$
$$\left(\phi_{ij}(s,t) = \mathrm{P}\left[Z_t = j | Z_s = i\right]\right).$$

They are obtained solving the following evolution equations:

$$\phi_{ij}(t) \;=\; \delta_{ij}(1 - H_i(t)) + \sum_{\beta=1}^{m}\sum_{\vartheta=1}^{t} b_{i\beta}(\vartheta)\phi_{\beta j}(t-\vartheta)$$

$$\left(\phi_{ij}(s,t) \;=\; \delta_{ij}(1 - H_i(s,t)) + \sum_{\beta=1}^{m}\sum_{\vartheta=1}^{t} b_{i\beta}(s,\vartheta)\phi_{\beta j}(\vartheta,t)\right)$$

where $\delta_{ij}$ represents the Kronecker symbol.

## 3 The discrete time homogeneous and non-homogeneous Markov and semi-Markov reward processes

Now a reward structure will be introduced, this structure is connected with the Z process. The reward process, both in Markov and semi-Markov cases, can be considered a class of stochastic processes in which, depending on the hypotheses, the evolution equation varies. In non-homogeneous case the rewards can depend also on the time of entrance in the state. Furthermore the non-homogeneity can involve the interest law in the sense that the interest rate can depend on the time of beginning of the operation and the time in which the operation ends (non-homogeneous time interest rate laws). This fact implies that in the non-homogeneous environment should be considered more cases than in homogeneous one. There are permanence rewards and transition rewards. In the literature they are also called respectively rate rewards and impulse rewards;; the first represents the reward given for the permanence in a state and the second the one paid because of a transition. The reward processes can be discounted or non-discounted. We are dealing with financial phenomena and we will present only the discounted cases. As already we told, there many different evolution equations (in non-homogeneous case more than three hundreds) but we will present the general cases. We will distinguish only between the immediate and the due cases. In the first the instalment is paid at the end of each period in the second at the beginning. This distinction seems to be trivial but from computational point of view it assumes great relevance. This time we present before the Markov relation that in the immediate case has the following structure:

$$V_i^{(n)} = V_i^{(n-1)} + \nu(n) \cdot \sum_{k=1}^{m} p_{ik}^{(n-1)} \left( (1 - swpe)\psi_k(n) + \sum_{j=1}^{m} p_{kj} \left( \gamma_{kj}(n) + swpe \cdot \psi_{kj}(n) \right) \right),$$

$$\left( V_i^{(n)}(s) = V_i^{(n-1)}(s) + \nu(s, s+n) \sum_{k=1}^{m} p_{ik}^{(n-1)}(s) \cdot \left( (1 - swpe)\psi_k(s, s+n) + \sum_{j=1}^{m} p_{kj}(s+n) \left( \gamma_{kj}(s, s+n) + swpe \cdot \psi_{kj}(s, s+n) \right) \right), \right)$$

where $V_i^{(n)}$ $\left( V_i^{(n)}(s) \right)$ represents the mean present value of all the rewards paid from 0 to $n$ ( $s$ to $s+n$) and $\nu(s)$ $(\nu(s, s+n))$ the corresponding discount factor. Furthermore *swpe* represents a variable that will have value 1 if the permanence rewards depend on the next transition and 0 if they do not depend on the transition. In Markov due case we have the following relation:

$$\ddot{V}_i^{(n)} = \ddot{V}_i^{(n-1)} + \nu(n) \sum_{k=1}^{m} p_{ik}^{(n-1)} \sum_{j=1}^{m} p_{kj} \gamma_{kj}(n) + \nu(n-1) \left( (1 - swpe) \sum_{k=1}^{m} p_{ik}^{(n-1)} \psi_k(n-1) + swpe \sum_{k=1}^{m} p_{ik}^{(n-2)} \sum_{j=1}^{m} p_{kj} \psi_{kj}(n-1) \right)$$

$$\left( \ddot{V}_i^{(n)} = \ddot{V}_i^{(n-1)} + \nu(s, s+n) \sum_{k=1}^{m} p_{ik}^{(n-1)} \sum_{j=1}^{m} p_{kj} \gamma_{kj}(s, s+n) + \nu(s, s+n-1) \cdot \left( (1 - swpe) \sum_{k=1}^{m} p_{ik}^{(n-1)} \psi_k(s, s+n-1) + swpe \sum_{k=1}^{m} p_{ik}^{(n-2)} \sum_{j=1}^{m} p_{kj} \psi_{kj}(s, s+n-1) \right) \right)$$

In the semi-Markov immediate case the relation is the following:

$$V_i(t) = (1 - swpe)(1 - H_i(t)) \sum_{\tau=1}^{t} \psi_i(\tau)\nu(\tau) + swpe(1 - H_i(t)) \sum_{k=1}^{m} \varphi_{ik}(t) \sum_{\tau=1}^{t} \psi_{ik}(\tau)\nu(\tau)$$
$$+ \sum_{k=1}^{m}\sum_{\vartheta=1}^{t} b_{ik}(\vartheta) \sum_{\tau=1}^{\vartheta} \psi_{ik}(\tau)\nu(\tau) + \sum_{k=1}^{m}\sum_{\vartheta=1}^{t} b_{ik}(\vartheta)\gamma_{ik}(\vartheta)\nu(\vartheta) + \sum_{k=1}^{m}\sum_{\vartheta=1}^{t} b_{ik}(\vartheta)V_k(t-\vartheta)\nu(\vartheta)$$

$$\left(\begin{array}{l} V_i(s,t) = (1 - swpe)(1 - H_i(s,t)) \displaystyle\sum_{\tau=s+1}^{t} \psi_i(s,\tau)\nu(s,\tau) \\ +swpe(1 - H_i(s,t)) \displaystyle\sum_{k=1}^{m} \varphi_{ik}(s,t) \sum_{\tau=s+1}^{t} \psi_{ik}(s,\tau)\nu(s,\tau) \\ +\displaystyle\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t} b_{ik}(s,\vartheta) \sum_{\tau=s+1}^{\vartheta} \psi_{ik}(s,\tau)\nu(s,\tau) \\ +\displaystyle\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t} b_{ik}(s,\vartheta)\gamma_{ik}(s,\vartheta)\nu(s,\vartheta) + \sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t} b_{ik}(s,\vartheta)V_k(\vartheta,t)\nu(s,\vartheta) \end{array}\right)$$

where:

$$\varphi_{ij}(t) = \frac{p_{ij} - Q_{ij}(t)}{1 - H_i(t)} \quad \left(\varphi_{ij}(s,t) = \frac{p_{ij}(s) - Q_{ij}(s,t)}{1 - H_i(s,t)}\right)$$

In the due case we have:

$$\ddot{V}_i(t) = (1 - swpe)(1 - H_i(t)) \sum_{\tau=0}^{t-1} \psi_i(\tau)\nu(\tau) + swpe(1 - H_i(t)) \sum_{k=1}^{m}\sum_{\tau=0}^{t-1} \varphi_{ik}(t)\psi_{ik}(\tau)\nu(\tau)$$
$$+ \sum_{k=1}^{m}\sum_{\vartheta=1}^{t} b_{ik}(\vartheta) \sum_{\tau=0}^{\vartheta-1} \psi_{ik}(\tau)\nu(\tau) + \sum_{k=1}^{m}\sum_{\vartheta=1}^{t} \nu(\vartheta)b_{ik}(\vartheta)\gamma_{ik}(\vartheta) + \sum_{k=1}^{m}\sum_{\vartheta=1}^{t} \nu(\vartheta-1)b_{ik}(\vartheta)\ddot{V}_k(t-\vartheta)$$

$$\left(\begin{array}{l} \ddot{V}_i(s,t) = \displaystyle\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t} b_{ik}(s,\vartheta) \sum_{\tau=s}^{\vartheta-1} \psi_{ik}(s,\tau)\nu(s,\tau) \\ +\displaystyle\sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t} \nu(s,\vartheta)b_{ik}(s,\vartheta)\gamma_{ik}(s,\vartheta) + \sum_{k=1}^{m}\sum_{\vartheta=s+1}^{t} \nu(s,\vartheta-1)b_{ik}(s,\vartheta)\ddot{V}_k(\vartheta,t) \\ +(1 - swpe)(1 - H_i(s,t)) \displaystyle\sum_{\tau=s}^{t-1} \psi_i(s,\tau)\nu(s,\tau) \\ +swpe(1 - H_i(s,t)) \displaystyle\sum_{k=1}^{m}\sum_{\tau=s}^{t-1} \varphi_{ik}(s,t)\psi_{ik}(s,\tau)\nu(s,\tau). \end{array}\right)$$

## 4 Stochastic annuities

**Definition 1** *Let:*

$$E = \{1, 2, \ldots, m\}$$

*be the states of a system and A, B two persons. Furthermore, let*

$$\mathbf{S} = \{S_1, S_2, \ldots, S_m\}\,,\ S_i \in \mathbb{R}$$

*be sums. The sums* **S** *represent the instalments of the annuity. The instalment* $S_i$ *will be paid or received from it A to it B if the system is in the state* $i$*. The instalment will be given for each period of the contractual time horizon. We say that this financial operation is a discrete time homogeneous constant stochastic annuity if:*

*i) the transitions among the states are governed by a homogeneous discrete time Markov Chain* $\mathbf{P} = [p_{ij}]$

*ii) when there is a transition from* $i$ *to* $j$ *it is possible that is paid or received a sum* $\gamma_{ij}$*.*

*The sums* $\gamma_{ij}$ *are named transition payments.*

- The annuity will be respectively *immediate* if the payments of the $\psi_i$ are scheduled at the *end* of the period and *due* at the *beginning*.

- The annuity is *non-homogenous* if the Markov chain is non-homogeneous. In this case it results $\mathbf{P}(t) = [p_{ij}(t)]$

- The annuity can be *variable* if the instalments and/or the transition payments change during the time horizon. In the non-homogeneous case the sums paid or received can vary also in function of the *starting* time of the financial operation.

It is useful to report the following

**Remark 1** *If there is a single state then the discrete time stochastic annuity corresponds to the usual concept of discrete time annuity.*

**Remark 2** *The concepts of homogeneous and non-homogeneous discrete time stochastic annuity correspond respectively to the ones of discrete time homogeneous and non-homogeneous Markov reward processes.*

**Definition 2** *Under the same condition of* **Definition 1** *we have the generalized case if the statement i) becomes:*

*i') the transitions among the states are governed by a homogeneous discrete time semi-Markov Chain with kernel* $\mathbf{Q}(t) = [Q_{ij}(t)]$

- The generalized stochastic annuity is *non-homogenous* if the semi-Markov chain is non-homogeneous, and the kernel becomes $\mathbf{Q}(s,t) = [Q_{ij}(s,t)]$.

**Remark 3** *The concepts of homogeneous and non-homogeneous discrete time generalized stochastic annuity correspond respectively to the ones of discrete time homogeneous and non-homogeneous semi-Markov reward processes.*

## 5 Multiple state insurance models and discrete time Markov and semi-Markov reward processes

The definition of multiple state insurance models corresponds with the definition of graph see [Haberman and Pitacco, 1999]. A multiple state model corresponds with a graph that describes the transitions among the states of the considered problem. The transition matrix describes the multiple state insurance models in the homogeneous case. In non homogeneous case a sequence of transition matrices describes the multiple state model. Premiums and benefits can be considered as rewards. The evolution of a general multiple state insurance model could be studied by means of Markov or semi-Markov models under the property that future is function only of the present. As well known, in discrete time the Markov process has the property that the time interval between two subsequent transitions is always the same. In the semi-Markov case the time between two transitions is a random variable. Some times an insurance contract can be studied well by means of a Markov process, some times the Markov environment is necessary because the transition are scheduled at each period (there is no randomness in the transition times), i.e. motorcar insurance. But in general in insurance problems the semi-Markov environment fits better than Markov one. In fact in the most part of insurance contracts the time of transition is stochastic. It is clear that in this light a multiple state insurance problem should be dealt in a better way by semi-Markov models. The reward processes gives the possibility to take into account directly the benefits and premiums that are considered in the multiple state models. Furthermore, usually, the insurance models are non-homogeneous respect the age of the insured person. It could be possible to use continuous time semi-Markov processes see [CMIR12, 1991] to construct multiple state insurance models. The problem in this case is that the solution of evolution equation is a very difficult task and that the analytical solution, excluding few particular cases not useful in the real problems, is impossible to find. The way could be the numerical solution of the evolution equation. But as it was shown in [Janssen and Manca, 2001], the numerical discretization corresponds to the discrete time processes. Summarizing we think that the best way to solve the multiple state insurance problem under Markov hypotheses is given by the application of DTNHSMRWP. In some cases the Markov environment suffices or it is necessary. Usually the problem should be faced in non-homogeneous environment. To construct non-homogeneous Markov or semi-Markov chains it is necessary to have huge amount of data that some times are not available, in these cases homogeneous environment should be used. Now considering what we state in the previous section we can affirm that *each multiple state insurance model can be considered a stochastic or a generalized stochastic annuity* depending on the insurance contract to be modelled. This statement confirms the fact that insurance problems should be considered as a generalization of financial problems in which the stochastic aspects assume great relevance.

## 6 Conclusions

In the paper the description of discrete time homogeneous and non-homogeneous semi Markov processes was given. After the concepts of Markov and semi-Markov reward processes were presented. The definitions of stochastic annuity and generalized stochastic annuity have been presented. The strict relation between the annuities and the reward processes was outlined.

All the paper moved in a discrete time approach because the applications are more suitable in this environment.

The paper should be seen as a theoretic step of these topics, for this reason there are no applications. The applications were presented in some less general paper see [Janssen and Manca, 2004]. In a near future the authors hope to study in depth the applicative aspects of the concepts given in this paper.

## References

[CMIR12, 1991]CMIR12 (Continuous Mortality Investigation Report, number 12).,*The analysis of permanent health insurance data.* 1991 The Institute of Actuaries and the Faculty of Actuaries.

[De Dominicis *et al.*, 1986]De Dominicis, R, Manca, R Some new results on the transient behaviour of semi-Markov reward processes.*Methods of Operations Research*, 1986 54.

[Haberman and Pitacco, 1999]Haberman, S. and Pitacco, E., *"Actuarial models for disability Insurance"*, 1999, Chapman and Hall.

[Howard, 1971]Howard, R.,*Dynamic probabilistic systems*, 1971 vol I II, Wiley.

[Iosifescu Manu, 1972]Iosifescu Manu A., Non homogeneous semi-Markov processes,*Stud. Lere. Mat.* 1972 24, 529-533.

[Janssen, 1966]Janssen, J., Application des processus semi-markoviens à un probléme d'invalidité,*Bulletin de l'Association Royale des Actuaries Belges*, 1966 63, 35-52.

[Janssen and Manca, 2001]Janssen J. and Manca R., Numerical solution of non homogeneous semi-Markov processes in transient case.*Methodology and Computing in Applied Probability.* 2001 Kluwer, 3, 271-293.

[Janssen and Manca, 2004]Janssen, J. and Manca, R., Discrete Time Non-Homogeneous Semi-Markov Reward Processes, Generalized Stochastic Annuities and Multi-State Insurance Model. *Proc. of XXVIII AMASES* Modena 2004.

[Wolthuis, 2003]Wolthuis H.,*Life Insurance Mathematics (the Markovian Model)*, II edition, 2003 Peeters Publishers, Herent.