Homogeneous Backward Semi-Markov Reward Models for Insurance Contracts

Raimondo Manca[1], Dmitrii Silvestrov[2], and Fredrik Stenberg[2]

[1] Universitá di Roma La Sapienza
via del Castro Laurenziano, 9
00161 Roma Italy
(e-mail: raimondo.manca@uniroma1.it)

[2] Malardalen University
Box 883
SE-721 23 Vasteras Sweden (e-mail: dmitrii.silvestrov@mdh.se,
fredrik.stenberg@mdh.se

Abstract. Semi-Markov reward processes are a very important tool for the solution of insurance problems. In disability problems can assume great relevance the date of the disability accident. In fact the mortality probability of a disabled person of a given age is higher respect the one of a person of the same age that is healthy. But the difference decreases with the running of the time after the instant of the disability. By means of backward semi-Markov processes it is possible to take in account the duration of the disability for an insured person. In this paper is shown for the first time, at authors' knowing, how to apply backward semi-Markov reward processes in insurance environment. An application will be shown.
Keywords: backward semi-Markov processes, reward processes, disability insurance.

1 Introduction

Semi-Markov processes was first defined by [Levy, 1954] in the fifties. At the beginning their application was in engineering, mainly where the application were linked to ageing. The use of so called multiple state models have long been used in the actuarial world for dealing with disability and illness among other things, see for example the book by [Haberman and Pitacco, 1999]. These models can be described by the use of semi-Markov processes and semi-Markov reward processes. An insurance contract ensures the holder benefits in the future from some random event(s) occurring at some random moment(s). The holder of the insurance contract pays a premium for the contract. Denote the discounted cash flow that occurs between the counter parties as the discounted accumulated reward where both the premiums and benefits are considered to be rewards. When developing an insurance contract between the writer and receiver the following questions must be asked. How shall the reward structure of the contract be determined? The fee can depend on the individuals exposure to becoming disabled in different states, and the benefits can be of two types, either instant rewards associated with transition between states or permanence rewards associated with maintaining in a state. In time evolution of insurance problems it is necessary to consider two different kind of randomness. One is originated by the accumulation during the time of the premiums and benefits paid or received (the financial evolution); the other is given by the time of the state change of the insured person, usually in insurance problem the transition among the states are effected at a random time. A semi-Markov environment can naturally take into account of both the two random aspects. This property was outlined for example in [Janssen and Manca, 2003] and [Janssen and Manca, 2004]. Another problem in insurance mainly in disability is the fact that the probability to change state is function of the distance from the moment of the disability. For example the probability to die in a disabled person of a given age is higher respect the one of a person of the same age that is healthy. But the difference decreases with the running of the time. In this paper the authors will consider also this duration effect using a bacward homogeneous semi-Markov reward process. By means of semi-Markov reward both financial and transition time randomness will be considered. By means of the backward environment also the duration phenomenon can be taken into account. It is to remark that, at authors' knowing, it is the first time that this last problem is faced by means of SMP in insurance field.

2 Homogenous Model.

Given the probability space (Ω, F, P) consider a homogenous Markov renewal process (X_n, T_n), $T_0 \leq T_1 \leq T_2 \leq ...$. Let the stochastic process $X_n, n \in \mathbb{N}$ have state space $E = \{1, 2, ..., m\}$ representing the state at the n-th transition. Let T_n represent the random time of the n-th transition with state space $\mathbb{N}$. For the combined process (X_n, T_n) define $Q_{ij}(t), b_{ij}(t), S_i(t)$ as,

$$Q_{ij}(t) = P(X_{n+1} = j, T_{n+1} - T_n \leq t | X_n = i) \tag{1}$$

$$b_{ij}(t) = P(X_{n+1} = j, T_{n+1} - T_n = t | X_n = i) \tag{2}$$

$$S_i(t) = P(T_{n+1} - T_n \leq t | X_n = i). \tag{3}$$

We allow for $Q_{ii}(t) \neq 0$, $t = 1, 2, ...$, i.e., artificial jumps from state i to itself, this is due to that sometimes this possibility makes sense in insurance applications. Impose $Q_{ij}(0) = 0$ for all $i, j \in E$, i.e., no instantaneously jumps in our process. Obviously,

$$S_i(t) = \sum_j Q_{ij}(t) \tag{4}$$

and

$$b_{ij}(t) = Q_{ij}(t) - Q_{ij}(t-1). \tag{5}$$

It is well known that,

$$p_{ij} = \lim_{t\to\infty} Q_{ij}(t) \quad i,j \in E$$

where $\mathbf{P} = [p_{ij}]$ is the transition probability of the embedded Markov chain. The conditional distribution functions for waiting time in each state i is given the state subsequently is j is given by,

$$G_{ij}(t) = P(T_{n+1} - T_n \leq t | X_n = i, X_{n+1} = j) = \begin{cases} \frac{Q_{ij}(t)}{p_{ij}}, & \text{if } p_{ij} \neq 0, \\ 1, & \text{if } p_{ij} = 0. \end{cases}$$

Define $\kappa(t)$ in the following way,

$$\kappa(t) = t - \max_{T_n \leq t} \{T_n\}\,. \tag{6}$$

$\kappa(t)$ describes the time already spent in the current state at time t.

3 Homogenous Rewards

The notation of rewards is given by;

ψ_i, $\psi_i(\kappa(t))$, $\psi_i(\kappa(t), t)$ denotes the rewards that are given for the permanence in the i-th state. The first reward doesn't change with the time and the future transition. The second changes with the time spent in the state. The third changes with the time spent in the state and is function of $\kappa(t)$ and t. They represent the flows of annuity that is paid during the presence in state i.

γ_{ij}, $\gamma_{ij}(\kappa(t))$, $\gamma_{ij}(\kappa(t), t)$ denote the rewards that are given for the transition from the i-th state to the j-th one. The distinctions among the three impulse rewards is the same given previuosly for the permanence rewards.

We will in this paper focus on constant rewards but our result can be extended into the other cases on the expense of more notation and indexes.

Let $e^{-t\delta}$ denote the discount factor for t periods with common fixed intensity of interest rate δ. Let $\xi_{i,u}(s,t)$, $s \leq t$ denote the accumulated discounted reward from s excluding s up to and including t given that the at time s the process is at state $i \in E$ and the previous jump accursed u moments ago. Here we apply the convention that $\xi_{i,u}(t,t) = 0$ for all t.

Theorem 1 *The reward process $\xi_{i,u}(s,t)$ is homogenous*

$$\xi_{i,u}(s,t) \stackrel{d}{=} \xi_{i,u}(0, t-s) \quad \forall i, u, s, t. \tag{7}$$

if the underlying process is a homogenous semi-Markov process and if the rewards only depends on $\kappa(t)$.

Introduce $T_{i,u}$ with the following distribution,

$$P(T_{i,u} > t) = \frac{1 - S_i(t+u)}{1 - S_i(u)} \tag{8}$$

and

$$P(T_{i,u} = s, X_{i,u} = j) = \frac{b_{ij}(u+s)}{1 - S_i(u)}. \tag{9}$$

Then $T_{i,u}$ describes the time to the next jump given that the process already have visited the state i for u units of time and let $X_{i,u}$ denote the corresponding state we end up in after the jump.

Let us assume $u = 0$, and first find a recursive relation for $\xi_{i,0}(0,t)$. We will have to consider two cases, if no jump occurs before moment t, or if at least one jump occurs between moment 0 up to moment t. If we introduce the indicator variables for these events we fill find the following relationship for $\xi_{i,0}(0,t)$,

$$\begin{aligned} \xi_{i,0}(0,t) \stackrel{d}{=} {} & \chi(T_{i,0} > t) \sum_{s=1}^{t} \psi_i e^{-\delta s} \\ & + \sum_j \sum_{s=1}^{t} (\chi(T_{i,0} = s, X_{i,0} = j)(e^{-\delta s}\gamma_{ij}(s) + \sum_{u=1}^{s} \psi_i(s) e^{-\delta u})) \qquad (10) \\ & + \sum_j \sum_{s=1}^{t} (\chi(T_{i,0} = s, X_{i,0} = j) e^{-\delta s} \xi_{j,0}(0, t-s)) \quad i \in E, \ \ t = 1, 2, \ldots \end{aligned}$$

where $\xi_{j,0}(0, t-s)$ are independent of indicators $\chi(T_{i,0} = s, X_{i,0} = j)$ and $\chi(T_{i,0} > t)$. The first term represents the discounted reward we receive at moment u to jump from state i to state j, the second term is due to the fact that the process restarts and is Markov at the moment of jump together with the assumption of homogeneities. The last term consists of the rewards we receive for the presence in state i between the moment 0 and u. This defines a closed system of equations which recursively can be solved.

To simplify the expression we can introduce some notation,

$$a_i(t) = \sum_{s=1}^{t} \psi_i(s) e^{-\delta s} \tag{11}$$

$$\tilde{a}_{ij}(t) = a_i(t) + e^{-\delta t}\gamma_{ij}(t). \tag{12}$$

Here $a_i(t)$ corresponds to the discounted accumulated reward for persistence in state i for t moments of time and $\tilde{a}_{ij}(t)$ the discounted accumulated reward for the persistence in state i for t moments of time plus the discounted instant reward for transition from state i to j at time t.

Then,

$$\xi_{i,0}(0,t) \stackrel{d}{=} \chi(T_{i,0} > t)a_i(t) + \sum_j \sum_{s=1}^{t} \chi(T_{i,0} = s, X_{i,0} = j)\tilde{a}_{ij}(s)$$
$$+ \sum_j \sum_{s=1}^{t} (\chi(T_{i,0} = s, X_{i,0} = j)e^{-\delta s}\xi_{j,0}(0, t-s) \quad i \in E, \quad t = 1, 2, ...$$

In the case $u \neq 0$, i.e., if we are interested in finding the accumulated reward toward a moment in time not associated with a jump;

$$\xi_{i,u}(0,t) \stackrel{d}{=} \chi(T_{i,u} > t)a_i(t) + \sum_j \sum_{s=1}^{t} \chi(T_{i,u} = s, X_{i,u} = j)\tilde{a}_{ij}(s)$$
$$+ \sum_j \sum_{s=1}^{t} \chi(T_{i,u} = s, X_{i,u} = j)e^{-\delta s}\xi_{j,0}(0, t-s) \quad i \in E, \quad t = 1, 2, ...$$

The only difference from the previous expressions is that we have to remember that our first jump-time depends on u, i.e., our surjeon time in the initial state is at least $u + 1$.

The first moment can now be calculated using these relationships, first consider the case $u = 0$,

$$\begin{aligned} E[\xi_{i,0}(0,t)] &= E[\chi(T_{i,0} > t)]a_i(t) + \sum_j \sum_{s=1}^{t} E[\chi(T_{i,0} = s, X_{i,0} = j)]\tilde{a}_{ij}(s) \\ &+ \sum_j \sum_{s=1}^{t} E[\chi(T_{i,0} = s, X_{i,u} = j)]E[\xi_{j,0}(0, t-s)]e^{-\delta s} \\ &= (1 - S_i(t))a_i(t) + \sum_j \sum_{s=1}^{t} b_{ij}(s)\tilde{a}_{ij}(s) \qquad (13) \\ &+ \sum_j \sum_{s=1}^{t} b_{ij}(s)E[\xi_{j,0}(0, t-s)]e^{-\delta s} \quad i \in E, \quad t = 1, 2, ... \end{aligned}$$

which follows from independence mentioned earlier. This set of equations can recursively be solved. Let $V_i(t) = E[\xi_{i,0}(0,t)], i \in E, t = 1, 2, ...$ then

$$
\begin{aligned}
&V_i(0) = 0 && \forall i \in E\\
&V_i(1) = (1 - S_i(1))a_i(1) + \sum_j b_{ij}(1)\tilde{a}_{ij}(1) && \forall i \in E\\
&V_i(2) = (1 - S_i(2))a_i(2) + \sum_j b_{ij}(1)\tilde{a}_{ij}(1) + \sum_j b_{ij}(2)\tilde{a}_{ij}(2)\\
&\qquad + \sum_j e^{-\delta} b_{ij}(1)V_j(1) \quad i \in E. && (14)
\end{aligned}
$$

And in general,

$$
V_i(t) = (1 - S_i(t))a_i(t) + \sum_j \sum_{s=1}^{t} b_{ij}(s)\tilde{a}_{ij}(s) + \sum_j \sum_{s=1}^{t} e^{-\delta s} b_{ij}(s)V_j(t-s)
$$
$$
\forall i \in E
$$

The values of $S_i(t),\ a_i(t),\ \tilde{a}_{ij}(t)$ are known and the only unknown parameters are $V_i(t)$. Above we see how we can recursively determine $V_i(t)$ by recursively solving $V_j(1), V_j(2), ..., V_j(t-1)$ for all $j \in E$.

And in the general case with $u \neq 0$,

$$
\begin{aligned}
E[\xi_{i,u}(0,t)] &= E[\chi(T_{i,u} > t)]a_i(t) + \sum_j \sum_{s=1}^{t} E[\chi(T_{i,u} = s, X_{i,u} = j)]\tilde{a}_{ij}(s)\\
&\quad + \sum_j \sum_{s=1}^{t} E[\chi(T_{i,u} = s, X_{i,u} = j)]E[\xi_{j,0}(0, t-s)]e^{-\delta s}\\
&= \frac{1 - S_i(t+u)}{1 - S_i(u)} a_i(t) + \sum_j \sum_{s=1}^{t} \frac{b_{ij}(u+s)}{1 - S_i(u)} \tilde{a}_{ij}(s) \qquad (15)\\
&\quad + \sum_j \sum_{s=1}^{t} \frac{b_{ij}(u+s)}{1 - S_i(u)} E[\xi_{j,0}(0, t-s)]e^{-\delta s} \quad i \in E,\ \ t = 1, 2, ...
\end{aligned}
$$

Note here that its enough to determine all $E[\xi_{j,0}(0,s)]$ for all $j \in E, s = 0, 1, ..., t-1$ to determine $E[\xi_{i,u}(0,t)]$. We are thereby back to our basic case $u = 0$.

4 Disability

In the papers by [Janssen and Manca, 2003]and [Janssen *et al.*, 2004] it is shown how to apply continuous time semi-Markov reward processes in multiple life insurance. In the paper by Blasi et al (2004), a real case study using real disability data is given. We will extend the example given in this paper using the backward homogeneous semi-Markov reward process that can take into account the duration of disability.

The model is a 5-state model. The considered states are the following:

states	disability degree	reward
1	$[0, .1)$	1000
2	$[.1, .3)$	1500
3	$[.3, .5)$	2000
4	$[.5, .7)$	2500
5	$[.7, 1]$	3000

The data gives the disability history of 840 persons that had silicosis problems and that live in Campania, a region in Italy. The reward is given to construct the example, it represents the money amount that is paid for each time period to the disable in function of its degree of illness. The transition occurs after a doctor visit that can be seen as the check to decide in which state the disable person is in. This gives naturally an example where virtual transitions are possible.

To be able to apply the technique developed in this paper for homogenous semi-Markov processes, we must first construct the embedded Markov-chain. The transition matrix is constructed from real data and is reported in the following table.

	0-10	10-30	30-50	50-70	70-100
0-10	0	1	0	0	0
10-30	0	0.811	0.180	0.005	0.004
30-50	0	0.017	0.75	0.21	0.02
50-70	0	0.023	0.03	0.72	0.22
70-100	0	0	0	0	1

Next step is to construct the matrix valued waiting time distribution $G(t)$.

To show the difference due to the introduction of the backward process the results with $u = 0$ (that means that the person entered in the state i when we begin the study of the system) and with $u = 2$ are reported.

s	0-10	10-30	30-50	50-70	70-100
1	970,87	1456,31	1941,74	2427,18	2912,62
2	1913,47	2876,75	3831,08	4780,34	5740,40
3	2993,34	4278,51	5684,64	7081,26	8485,83
4	4283,87	5657,51	7510,09	9330,86	11151,29
5	5547,74	7015,79	9310,84	11536,94	13739,12
6	6792,15	8352,65	11078,77	13684,65	16251,57
7	8015,84	9667,20	12817,61	15778,89	18690,84
8	9219,65	10959,30	14522,48	17822,21	21059,07
9	10403,01	12229,95	16193,96	19816,49	23358,32
10	11565,50	13479,60	17830,89	21759,34	25590,60

mean total reward with $u = 0$

s	0-10	10-30	30-50	50-70	70-100
1	970,87	1456,31	1941,74	2427,18	2912,62
2	2346,91	2904,34	3880,52	4827,55	5740,40
3	3688,40	4340,42	5812,47	7207,04	8485,83
4	5009,97	5759,51	7714,30	9527,12	11151,29
5	6308,27	7159,74	9590,50	11792,99	13739,12
6	6792,15	8352,65	11078,77	13684,65	16251,57
7	8840,21	9892,87	13237,77	16171,26	18690,84
8	10072,70	11227,54	15006,35	18279,73	21059,07

mean total reward with $u = 2$

Few words to describe the results. We present an example only to show that taking into account the permanence into the state before the beginning of the study of the system changes the results. We did not change the transition probabilities changing the backward variable u. The different dead probability means different transition probability. But also without changing the probabilities the results were different. It is only to observe that the last state is absorbing and from (15) it follows that the results do not change. Furthermore the payments of the first year are always the same because they are equal to the corresponding first discounted rewards as it was proved in [Janssen and Manca, 2005].

5 Conclusions

In this paper a first step for the application of the backward semi-Markov reward in insurance field was done. In future works the authors would generalize this approach in non homogeneous environment. Reward processes represent the first moment of the total revenues that are given in a stochastic financial operation. The author would also find models and algorithms useful to compute the higher moments.

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