Credit risk migration semi-Markov models: a reliability approach.

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Abstract. Credit risk problem is one of the most important financial topics in this period because of the Basel II rules. In 1997 a seminal paper Jarrow Lando and Turnbull showed that this problem could be approached by means of a Markov chain tool. Subsequently in many papers it was shown that the Markov approach can give some problems, more precisely: In some previous papers the authors showed how it is possible by means of a reliability semi-Markov approach to solve the three problems. In this paper will be summarized the results obtained by the authors to give a complete overview of the proposed approach.
Keywords: Credit risk, semi-Markov, reliability.

1 Introduction

Homogeneous semi-Markov processes were defined in the fifties in [Levy, 1954]. Non-homogeneous semi-Markov processes were defined in [Iosifescu Manu, 1972]. A detailed theoretical analysis of semi-Markov processes was given in [Howard, 1971]. The importance of the Engineering applications of this kind of processes is highlighted in this book. As specified in [Howard, 1971] and more recently in [Limnios and Oprisan, 2000] book, one of the most important applications of semi-Markov processes is in reliability of mechanical systems. Putting the hypothesis that the next transition depends only on the last one (the future depends only on the present) the problem can be faced by means of Markov processes. In discrete time Markov chain environment the time transition is given. But in the reality, the transition between two states in a mechanical system usually happens after a random duration. This is the reason why the semi-Markov environment fits better than the Markov one in reliability problems. Another relevant phenomenon in the time evolution of a system can be the system age. The introduction of non-homogeneity gives the possibility to take into account this problem. All the highlighted aspects can be faced using non-homogeneous semi-Markov models. In the paper [Blasi *et al.*, 2003] how it is possible to apply non-homogeneous semi-Markov processes in reliability problems is described. Credit risk problem is one of the most important problems that are faced in the financial literature. Fundamentally it consists in computing the default probability of a firm that do a debt. The literature on this topic is very wide, but the interested lector can refer to the [Duffie and Singleton, 2003] book. Big interest in this field is given to the firms that issue bonds. For the credit risk evaluation there are international organisations, Fitch, Moodys and Standard & Poors, that give different ranks to the examined firms. At each firm is given a "rating" that is a vote to the "reliability" on the capacity to reimburse the debt. The rating level changes in the time and one way to follow the time evolution of ratings is by means of Markov processes [Jarrow *et al.*, 1997]. In this environment Markov models are called "migration models". Other papers, see for example [Nickell *et al.*, 2000], followed this approach working mainly on the generation of transition matrix. In some papers the problem of the unfitting of Markov process in credit risk environment was outlined, see [Carty and Fons, 1994], [Nickell *et al.*, 2000]. The problems of non-markovianity that are highlighted mainly are the following:

i - the duration inside a state. The probability to change rating depends on the time that a firm remains in the same rating [Carty and Fons, 1994];

ii - the time dependence of the rating evaluation (aging). This means that in general the rating evaluation depends on the time in which is done, see [Nickell *et al.*, 2000] The rating evaluation done at time t generally is different from the one done at time s, if $s \neq t$;

iii - the dependence of the new rating on the previous ones, not only on the last evaluated, [Carty and Fons, 1994], [Nickell *et al.*, 2000].

The first problem can be well solved by means of semi-Markov processes (SMP). In fact in SMP the transition probabilities are function of the waiting time spent in a state of the system. The second problem can be faced in a general approach by means of a non-homogeneous environment. The third effect exists in the downward cases but not in the upward ratings. More precisely if a firm got a lower rating then has a higher probability that the next rating will be lower than the preceding one. The first two are automatically solved applying the non-homogeneous semi-Markov environment. The third problem is solved increasing the number of states to differentiate the case in which the system arrives in a state from a lower or a higher rating evaluation. In a previous article [D'Amico *et al.*, 2003] presented a model based on the homogeneous semi-Markov processes (HSMP) in a reliability environment. The duration problem was fully solved for the first time, at authors knowing, in that paper. The other two credit risk problems were not faced. A second paper [D'Amico *et al.*, 2004a] presenting a non-homogeneous semi-Markov process (NHSMP) model takes into accounts the duration and the aging problem. In a third paper [D'Amico *et al.*, 2004b] also the third problem was solved. The non-homogeneous semi-Markov reliability model, presented together the homogeneous one in [Blasi *et al.*, 2003], will be applied, to solve the credit risk problem.

This paper will present a summary of the three papers and will expose the approach that was made to solve the Markov migration problems. The next part will present a short description of NHSMP. After this the reliability non-homogeneous semi-Markov model will be shown. In the successive paragraph the relation between the reliability model and the credit risk problem will be described. The model enlarges the number of states in this way the downward problem can be solved.

2 Non-homogeneous semi-Markov processes

In this part the NHSMP will be described; we follow the notation given in [Janssen and Manca, 2005]. First the stochastic process is defined. In SMP environment two random variables (r.v.) run together. J_n $n \in \mathbb{N}$ with state space $I = \{1, 2, \ldots, m\}$ represents the state at the n-th transition. T_n $n \in \mathbb{N}$ with state space equal to $\mathbb{R}^+$ represents the time of the n-th transition,

$$J_n : \Omega \rightarrow I \;\; T_n : \Omega \rightarrow \mathbb{R}^+.$$

We suppose that the process (J_n, T_n) is a non-homogeneous markovian renewal process. The kernel $\mathbf{Q} = [Q_{ij}(s,t)]$ associated to the process is defined in the following way:

$$Q_{ij}(s,t) = \mathrm{P}[J_{n+1} = j, T_{n+1} \leq t | J_n = j, T_n = s]$$

and it results:

$$p_{ij}(s) = \lim_{t \to \infty} Q_{ij}(s,t), \;\; i, j \in I, s, t \in \mathbb{R}^+, s \leq t$$

where $\mathbf{P}(s) = [p_{ij}(s)]$ is the transition matrix of the embedded non-homogeneous Markov chain in the process. Furthermore it is necessary to introduce the probability that process will leave the state i from the time s up to the time t:

$$S_i(s,t) = \mathrm{P}[T_{n+1} \leq t | J_n = j, T_n = s]$$

Obviously it results that:

$$S_i(s,t) = \sum_{j=1}^{m} Q_{ij}(s,t)$$

Now it is possible to define the distribution function of the waiting time in each state i, given that the state successively occupied is known:

$$G_{ij}(s,t) = \mathrm{P}[T_{n+1} \leq t | J_n = j, J_{n+1} = j, T_n = s]$$

Obviously the related probabilities can be obtained by means of the following formula:

$$G_{ij}(s,t) \;=\; \begin{cases} Q_{ij}(s,t)/p_{ij}(s) & \text{if } p_{ij}(s) \neq 0 \\ 1 & \text{if } p_{ij}(s) = 0 \end{cases}$$

The main difference between a continuous time non-homogeneous Markov process and a NHSMP is in the increasing distribution functions $G_{ij}(s,t)$. In Markov environment this function has to be a negative exponential function. Instead in the semi-Markov case the distribution functions $G_{ij}(s,t)$ can be of any type. If we apply the semi-Markov model in the credit risk environment we can take into account, by means of the $G_{ij}(s,t)$ the problem given by the duration of the rating inside the states. Now the NHSMP $Z = (Z_t,\, t \in \mathbb{R}^+)$ can be defined. It represents, for each waiting time, the state occupied by the process. The transition probabilities are defined in the following way:

$$\phi_{ij}(s,t) = \mathrm{P}[Z_t = j | Z_s = i]$$

They are obtained solving the following evolution equations:

$$\phi_{ij}(s,t) \;=\; \delta_{ij}(1 - S_i(s,t)) + \sum_{\beta=1}^{m} \int_s^t \dot{Q}_{i\beta}(s,\vartheta)\phi_{\beta j}(\vartheta,t)d\vartheta \tag{1}$$

where δ_{ij} represents the Kronecker symbol. The first part of relation (1)

$$\delta_{ij}(1 - S_i(s,t)) \tag{2}$$

gives the probability that the system doesn't have transitions up to the time t given that it was in the state i at time s. The (2) formula in rating migration case represents the probability that the rating organisation doesn't give any new rating evaluation from the time s up to the time t. This part has sense if and only if $i = j$. In the second part

$$\sum_{\beta=1}^{m} \int_s^t \dot{Q}_{i\beta}(s,\vartheta)\phi_{\beta j}(\vartheta,t)d\vartheta$$

$\dot{Q}_{i\beta}(s,\vartheta)$ is the derivative at time ϑ of $Q_{i\beta}(s,\vartheta)$ and represents the probability intensity that the system was at time s in the state i and remained in this state up to the time ϑ and that it went to the state β just at time ϑ. After the transition the system will go to the state j following one of the possible trajectories that go from the state β at the time ϑ to the state j within the time t. In the credit risk environment it means that from the time s up the time ϑ the rating company doesn't give any other evaluation of the firm; at time ϑ the rating company gave the new rating β at the evaluating firm. After this the rating will arrive to the state j within the time t following one of the possible rating trajectories.

3 Non-homogeneous semi-Markov reliability model

There are a lot of semi-Markov models in reliability theory see for example [Limnios and Oprisan, 2000]. The non-homogeneous case was presented in [Blasi *et al.*, 2003]. Let us consider a reliability system S that can be at every time t in one of the states of $I = \{1, \ldots, m\}$. The stochastic process of the successive states of S is $Z = \{Z(t),\ t \geq 0\}$. The state set is partitioned into sets U and D, so that:

$$I = U \cup D,\ \ \emptyset = U \cap D,\ \ U \neq \emptyset,\ \ U \neq I$$

The subset U contains all "good" states in which the system is working and subset D all "bad" states in which the system is not working well or is failed. The classical indicators used in reliability theory are the following ones:
(i) *the non-homogeneous reliability function R* giving the probability that the system was always working from time s to time t:

$$R(s,t) = P\left[Z(u) \in U : \forall u \in (s,t]\right] \tag{3}$$

(ii) *the point wise non-homogeneous availability function A* giving the probability that the system is working on time t whatever happens on $(s,t]$:

$$A(s,t) = P\left[Z(t) \in U\right], \tag{4}$$

(iii) *the non-homogeneous maintainability function M* giving the probability that the system will leave the set D within the time t being in D at time s:

$$M(s,t) = 1 - P\left[Z(u) \in D,\ \ \forall u \in (s,t]\right]. \tag{5}$$

It is shown in [Blasi *et al.*, 2003] that these three probabilities can be computed in the following way if the process is a non-homogeneous semi-Markov process of kernel $\mathbf{Q}$.
(i) *the point wise availability function A_i* given that $Z_s = i$.

$$A_i(s,t) = \sum_{j \in U} \phi_{ij}(s,t) \tag{6}$$

(ii) *the reliability function R_i* given that $Z_s = i$. To compute these probabilities all the states of the subset D are changed in absorbing states. $R_i(s,t)$ is given by solving the evolution equation of HSMP but now with the embedded Markov chain having:

$$p_{ij}(s) = \delta_{ij} \ \text{ if } \ i \in D$$

The related formula will be:

$$R_i(s,t) = \sum_{j \in U} \phi^r_{ij}(s,t) \tag{7}$$

where $\phi^r_{ij}(s,t)$ is the solution of equation (1) with all the states in D that are absorbing;
(iii) *the maintainability function* M_i given that $Z_s = i$:
in this case all the states of the subset U are changed in absorbing states. $M_i(s,t)$ is given by solving the evolution equation of HSMP with the embeded Markov chain having:

$$p_{ij}(s) = \delta_{ij} \text{ if } i \in U.$$

The related formula will be:

$$M_i(s,t) = \sum_{j \in U} \phi^m_{ij}(s,t) \tag{8}$$

where $\phi^m_{ij}(s,t)$ is the solution of equation (1) with all the states in U that are absorbing.

4 Non-homogeneous semi-Markov reliability credit risk model

The credit risk problem can be situated in the reliability environment. The rating process, done by the rating agency, gives a reliability degree of a firm bond. In the Standard & Poors case there are the 8 different classes of rating that means to have the following set of states:

$$I = \{\text{AAA, AA, A, BBB, BB, B, CCC, D}\}$$

To take into account the downward problem we introduce other 6 states. The set of the states becomes the following:

$$I = \{\text{AAA, AA,AA - , A,A - , BBB,BBB - , BB,BB - , B,B - , CCC,CCC - , D}\}$$

For example the state BBB is divided in BBB and BBB-. The system will be in the state BBB if it arrived from a lower rating, instead it will be in the state BBB- if it arrived in the state from a better rating (a downward transition). It is also possible to suppose that if there is a virtual transition than if the system is in the BBB- state it will go to the BBB state.

The first 13 states are working states (good states) and the last one is the only bad state. The two subsets are the following:

$$U = \{\text{AAA, AA,AA - , A,A - , BBB,BBB - , BB,BB - , B,B - , CCC,CCC - }\}$$
$$D = \{\text{ D}\}$$

In this case the maintainability function M doesn't have sense because the default state D is absorbing and once that the system went in this state

it is not possible to leave it. Furthermore the fact that the only bad state is an absorbing state implies that the availability function A and the reliability function R correspond. In this case the reliability model is substantially simplified. In fact to get all the results that are relevant in the credit risk case it is enough to solve only once the system (2.1). Solving this system we will obtain the following results:
1) $\phi_{ij}(s,t)$, that represents the probabilities to be in the state j after a time t starting in the state i at time s. These results take into account the different probabilities to change state during the permanence of the system in the same state (duration problem) and the different probabilities to change state in function of the different time of evaluation (aging problem). The different probability values given for the two states that are obtained because of downward problem solve the third Markovian model problem.
2) $R_i(s,t) = A_i(s,t) = \sum_{j \in U} \phi_{ij}(s,t)$, that represents the probability that the system never goes in the default state from the time s up to the time t.
3) $1 - S_i(s,t)$, that represents the probability that from the time s up to the time t no one new rating evaluation was done for the firm.

Before to give another result that can be obtained in a SMP environment, we have to introduce the concept of the first transition after the time t. More precisely we suppose that the system at time s was in the state i. We know that with probability $1 - S_i(s,t)$ the system doesn't move from the state i. Under these hypotheses we would know the probability that the next transition will be to the state j. This probability will be denoted by $\varphi_{ij}(s,t)$. That has the following meaning:

$$\varphi_{ij}(s,t) = P\left[X_{n+1} = j | X_n = i,\ T_{n+1} > t, T_n = s\right] \tag{9}$$

This probability can be obtained by means of the following formula:

$$\varphi_{ij}(s,t) = \frac{p_{ij}(s) - Q_{ij}(s,t)}{1 - S_i(s,t)}$$

After the definition (9) by means of SMP it is possible to get the following result:
4) $\varphi_{ij}(s,t)$ represents the probability to get the rank j at next rating if the previous state was i and no one rating evaluation was done from the time s up to the time t. In this way, for example, if the transition to the default state is possible and if the system doesn't move from the time s up to the time t from the state i, we know the probability that in the next transition the system will go to the default state.

5 Conclusions

This paper summarizes the three theoretical step that the authors did to improve the so called migration models in the credit risk environment. The first step solved the problem of different probability transactions because of the time duration inside a rating state by means of introduction of SMP in credit risk environment. The second step, by means of non-homogeneity introduction in the SMP environment, gave the way to consider also the system time dependence problem. The third step solved the credit risk downward problem. The three models start from the idea that credit risk problem can be considered a special case of reliability problem and this idea allows the application of some non-homogeneous semi-Markov reliability results in the credit risk environment. The downward problem was solved enlarging the state number. Authors in the next future hope to be able to get data from rating companies. In this case they will apply to real data their credit risk models.

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