Valuing Credit Default Swap in a non-Homogeneous Semi-Markovian Rating-Based Model.

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Abstract. In this paper we use a discrete time non-homogeneous semi-Markov model for the rating evolution of the credit quality of a firm C and we determine the credit default swap spread for a contract between two parties, A and B that, respectively, sell and buy a protection about the failure of the firm C. We work in both the case of deterministic and stochastic recovery rate. We highlight the link between credit risk and reliability theory too.
Keywords: backward recurrence times processes, random recovery rate, reliability.

1 Introduction

The credit default swap (CDS) is a derivative that can be seen as default insurance on loans and bonds. These contracts are instruments that provide insurance against a particular company (that we will call company "C") defaulting on its debt. In this paper we present an evaluation procedure of credit default swap in a rating based model. We assume that the rating credit quality evolution of the company "C" that issue the bond follows a discrete time non-homogeneous semi-Markov process, so to consider the reference default risk we use the non-homogeneous semi-Markov reliability credit risk model [D'Amico *et al.*, 2004a]. In this way, how it is showed in [D'Amico *et al.*, 2004a], we solve all the non-markovianity problems highlighted by some empirical works in this area such [Carty and Fons, 1994] and [Nickell *et al.*, 2002].

We fix the credit default swap spread $U^*(s)$ imposing a fair game condition on the wealth balance equation for the swap contract. We compute $U^*(s)$ first considering a fixed recovery rate ρ and successively extending the computation to the case of a random recovery rate. Considering the non-homogeneity of the process we give the same definition of stochastic recovery rate as in [D'Amico *et al.*, 2004b] linking the random recovery rate in general on the last n states visited by the process first of the random default time τ_s.

In both the cases of deterministic and stochastic recovery rate, we express the price and the value of the swap as a function of the C's reliability.

2 The discrete time non-homogeneous semi-Markov reliability credit risk model

First of all we give some basic results on the theory of discrete time non-homogeneous semi-Markov processes. Let (Ω, F, P) be a probability space and let E be a finite state space. On our probability space we define two stochastic processes: $X_n : \Omega \longrightarrow E,\ \ T_n : \Omega \longrightarrow \mathbb{N}$.

X_n represents the state occupied at the n-th transition and T_n is the time of the n-th transition. The process (X, T) is a non-homogeneous Markov Renewal Process if $\forall i, j \in E$ and $\forall t \in \mathbb{N}$ the following condition holds:

$$P[X_{n+1}=j, T_{n+1}\leq t|\sigma(X_h, T_h), X_n=i, T_n=s, 0\leq h\leq n]=$$

$$P[X_{n+1}=j, T_{n+1}\leq t|X_n=i, T_n=s]\equiv Q_{ij}(s,t). \quad (1)$$

The transition matrix $P(s)$ of the non-homogeneous embedded Markov chain X_n is obtained as $p_{ij}(s) = \lim_{t\to\infty} Q_{ij}(s,t)\ \ \forall i, j \in E$.

We introduce also the following probabilities:

$$q_{ij}(s,t) = P[X_{n+1} = j, T_{n+1} = t|X_n = i, T_n = s], \quad (2)$$

$$H_i(t) = P[T_{n+1} \leq t|X_n = i, T_n = s], \quad (3)$$

Let $N(t) = sup\{n : T_n \leq t\}\ \forall t \in \mathbb{N}$; we define the non-homogeneous discrete time semi-Markov process $Z = (Z(t), t \in \mathbb{N})$ as $Z(t) = X_{N(t)}$, that represents, for each waiting time, the state occupied by the process.

We define, $\forall i, j \in E$, and $(s, t) \in \mathbb{N} \times \mathbb{N}$, the semi-Markov's transition probabilities as $\phi_{ij}(s,t) = P[Z(t) = j|Z(s) = i]$ satisfying the following system of equations:

$$\phi_{ij}(s,t) = \delta_{ij}(1 - H_i(s,t)) + \sum_{k\in E}\sum_{\tau=1}^{t} q_{ik}(s,\tau)\phi_{kj}(\tau,t). \quad (4)$$

At this time we explain briefly the non-homogeneous semi-Markov reliability credit risk model, see [D'Amico *et al.*, 2004a] to study in depth.

Let the state space E indicate the different rating classes that give a reliability degree of a firm bond. We partition this state space in two subset: $D = \{N + 1\}$ and $Up = \{1, 2, ..., N\}$, that we call respectively "Down" (default) and "Up" states. We assume that the set D is absorbing. The most important variable to compute is the reliability $R(s, \cdot)$ of the firm that is defined $\forall t \geq s$ as $R(s,t) = P[Z(u) \in Up,\ \forall u \in \{s, s + 1, ..., t\}]$. The reliability function $R_i(s,t)$ conditional on the starting state i at time s, is given by $R_i(s,t) = \sum_{j \in Up} \phi_{ij}(s,t)$, then solving the system of equations (4) (see [Blasi *et al.*, 2003]) and summing on the "Up" states we obtain the conditional reliability. Obviously $R(s,t) = \sum_{i \in Up} \sum_{j \in Up} \beta_i(s)\phi_{ij}(s,t)$ where $\beta(s) = (\beta_i(s))_{i \in E}$ denotes the random starting distribution at time s. In our model the reliability is equal to the availability, that give us the probability that the system is "Up" at the generic time t, because the only one defaulting state is absorbing.

3 The price and the value of the swap: the fixed recovery rate case

In this section we consider a CDS contract starting at time s with maturity T. We denote with $\tau_s = \inf\{t > s : Z(t) \in D\}$ and with v the deterministic discount factor. We write the wealth balance equation (w.b.e.) for the seller B of the protection about a failure of C that is given by:

$$\Delta W|_s^T = \sum_{i=s+1}^{T \wedge \tau_s} U(s) \cdot v^{i-s} - (100 - Y(T \wedge \tau_s)) \cdot v^{(T \wedge \tau_s)-s}. \tag{5}$$

The term $(\sum_{i=s+1}^{T \wedge \tau_s} U(s) \cdot v^{i-s})$ is the random discounted amount of money that B will obtain writing the CDS contract and $(100 - Y(T \wedge \tau_s)) \cdot v^{(T \wedge \tau_s)-s}$ is the potential loss in case of a C's default.

We assume that $Y(T \wedge \tau_s) = 100 \cdot \rho \cdot 1_{\{s < \tau_s \le T\}} + 100 \cdot 1_{\{\tau_s > T\}}$ where $\rho \in [0,1]$ is the deterministic recovery rate. This choice implies that the potential loss will be zero if there is no default up to time T whereas if a default occurs first of T the potential loss becomes a real loss equal to $100(1-\rho)$ discounted from default time to starting time s. Then the w.b.e. becomes:

$$\Delta W|_s^T = \sum_{i=1}^{T \wedge \tau_s} U(s) \cdot v^{i-s} - (100[1-\rho]) \cdot v^{(T \wedge \tau_s)-s} 1_{\{s < \tau_s \le T\}}.$$

Fixing the credit default swap spread $U(s)$ imposing a fair game condition so that the expectation of the w.b.e. is zero, we get in:

$$U^*(s) = \frac{(1-v)[100 \times (1-\rho)] E[v^{(T \wedge \tau_s)-s} 1_{\{s < \tau \le T\}}]}{(v) \times [1 - E[v^{(T \wedge \tau_s)-s}]]}. \tag{6}$$

Now having

$$E[v^{(T \wedge \tau_s)-s}] = \sum_{h=s+1}^{T} v^{h-s} \{R(s,h-1) - R(s,h)\} + v^{T-s} R(s,T) \tag{7}$$

$$E[v^{(T \wedge \tau_s)-s} 1_{\{\tau_s \le T\}}] = \sum_{h=s+1}^{T} v^{h-s} \{R(s,h-1) - R(s,h)\} \tag{8}$$

substituting in equation (6) we obtain:

$$U^*(s) = \frac{(1-v)[100 \times (1-\rho)][\sum_{h=s+1}^{T} v^{h-s}\{R(s,h-1) - R(s,h)\}]}{(v) \times [1 - \sum_{h=s+1}^{T} v^{h-s}\{R(s,h-1) - R(s,h)\} - v^{T-s}R(s,T)]}. \tag{9}$$

Now we turn our attention to the valuation procedure. The value of the swap at time t (conditional on no default first of time t) is given by the difference between the expected present value (at time t) of the future inflows minus the expected present value (at time t) of the future outflows. Let $V(s,t)$ the value of the swap and let $I(s,t) = \sum_{h=t+1}^{T \wedge \tau_s} U^*(s)v^{h-t}$ and $O(s,t) = 100(1-\rho)v^{(T\wedge\tau_s)-t}1_{\{s<\tau_s\leq T\}}$ be, respectively, the future inflows and the future outflows then by definition we have

$$V(s,t) = E[I(s,t) - O(s,t)|\tau_s > t] = E[I(s,t)|\tau_s > t] - E[O(s,t)|\tau_s > t]. \tag{10}$$

We obtain:

$$E[I(s,t)|\tau > t] = U^*(s)\Big\{\sum_{m=t+1}^{T} \frac{R(s,m-1) - R(s,m)}{R(s,t)}(\sum_{h=t+1}^{m} v^{h-t}) + \sum_{h=t+1}^{T} v^{h-t}\frac{R(s,T)}{R(s,t)}\Big\}. \tag{11}$$

$$E[O(s,t)|\tau_s > t] = 100(1-\rho)\Big\{\sum_{h=t+1}^{T} v^{h-t}\frac{R(s,h-1) - R(s,h)}{R(s,t)}\Big\}. \tag{12}$$

substituting in formula (10) we get the value of the swap at time t as a function of the reliability of the firm.

4 The price and the value of the swap: the random recovery rate case

In this section we extend our model considering a stochastic recovery rate ρ. [Berthault *et al.*, 2001] noted that the higher is the rating the lower is the loss in case of default. From this empirical evidence [Millossovich, 2002] linked the recovery rate to the last credit rating evaluation of the company first of the default time τ_s in a markovian time homogeneous environment. That extension was carried out enlarging the state space, considering multiple default classes, one for each possible recovery rate. [D'Amico *et al.*, 2004b] proposed a new way to allows for stochastic recovery rate that depends on the last (possibly n-last) rating evaluation, obtained first of the default time, without enlarging the state space E.

In this paper we use the same definition given in [D'Amico *et al.*, 2004b] being careful on the non-homogeneity of the rating process, so we define the one period stochastic recovery rate at time τ_s, "$\rho_1(\tau_s)$" in the following way:

$$\rho_1(\tau_s) = \begin{cases} r_j \text{ if } s < \tau_s \leq T \text{ and } Z(\tau_s - 1) = j, \forall j \neq D \\ 1 \text{ if } \tau_s > T > s \end{cases} \tag{13}$$

We proceed to compute the credit default swap spread $U^*(s)$ starting from equation (6) and imposing a fair game condition such that the expectation of the wealth balance equation is zero. In this case we get:

$$U_1^*(s) = \frac{(100[E[v^{(T\wedge\tau_s)-s}1_{\{s<\tau_s\leq T\}}] - E[\rho_1(\tau_s)v^{(T\wedge\tau_s)-s}1_{\{s<\tau_s\leq T\}}]]) \times (1-v)}{v \times (1 - E[v^{(T\wedge\tau_s)-s}])}. \quad (14)$$

The unique new component to evaluate is $E[\rho_1(\tau_s)v^{(T\wedge\tau_s)-s}1_{\{s<\tau_s\leq T\}}]$.

But $E[\rho_1(\tau_s)v^{T\wedge\tau_s}1_{\{s<\tau_s\leq T\}}] = E[E[\rho_1(\tau_s)v^{(T\wedge\tau_s)-s}1_{\{s<\tau_s\leq T\}}|\tau_s]]$ where

$$E[\rho_1(\tau_s)v^{(T\wedge\tau_s)-s}1_{\{s<\tau_s\leq T\}}|\tau_s] = \sum_{h=s+1}^{\infty} v^{(T\wedge h)-s}\rho_1(h)P[\tau_s = h]1_{\{h\leq T\}} =$$

$$\sum_{h=s+1}^{T} v^{h-s}\rho_1(h)P[\tau_s = h] = \sum_{h=s+1}^{T} v^{h-s}\rho_1(h)\{R(s,h-1) - R(s,h)\} \quad (15)$$

consequently $E[\rho_1(\tau_s)v^{T\wedge\tau_s-s}1_{\{s<\tau_s\leq T\}}] =$

$$\sum_{h=s+1}^{T} v^{h-s}\sum_{j\in Up} r_j P[Z(h-1)=j|Z(h)=D]\{R(s,h-1) - R(s,h)\} \quad (16)$$

To compute $P[Z(h-1) = j|Z(h) = D]$ we have to introduce the non-homogeneous discrete backward recurrence time process $B(t)$ defined as:

$$B(t) = \begin{cases} t + T_0 & \text{if } t < T_1 \\ t - T_{N(t)} & \text{if } t \geq T_1 \end{cases} \quad (17)$$

We know that the stochastic process $(Z(t), B(t))$ with values in $E \times \mathbb{N}$ is a markovian process and $\forall h \in \{1, 2, ...T\}$ and $j \in E$ conditioning on all possible values for $B(h-1)$ and from Bayes formula we have that

$$P[Z(h-1) = j|Z(h) = D] =$$

$$\frac{\sum_{l=0}^{h-1-s} P[Z(h)=D|Z(h-1)=j, B(h-1)=l]P[Z(h-1)=j, B(h-1)=l]}{\sum_{k\in Up}\sum_{l=0}^{h-1-s} P[Z(h)=D|Z(h-1)=k, B(h-1)=l]P[Z(h-1)=k, B(h-1)=l]} =$$

$$\frac{\sum_{i\in E}\beta_i(s)\sum_{l=0}^{h-1-s} L_{ij}(s,h-1,l)\Delta_{jD}(h-1,l,h)}{\sum_{i\in E}\beta_i(s)\sum_{k\in Up}\sum_{l=0}^{h-1-s} L_{ik}(s,h-1,l)\Delta_{kD}(h-1,l,h)} \quad (18)$$

where $\Delta_{ij}(h,l,t) = P[Z(t) = j|Z(h) = i, B(h) = l]$ and

$$L_{ij}(s,h,l) = P[Z(h)=j, B(h)=l|Z(s)=i, B(s)=0] = P_{(i,s)}[Z(h)=j, B(h)=l].$$

These probabilities can be computed from the knowledge of the semi-Markov kernel **Q** in fact we have, see [D'Amico *et al.*, 2004c], that

$$\Delta_{ij}(h,l,t)=\frac{\delta_{ij}(1-H_i(h-l,t))}{(1-H_i(h-l,h))}+\frac{1}{(1-H_i(h-l,h))}\sum_{k\in E}\sum_{m=h+1}^{t}q_{ik}(h-l,m)\phi_{kj}(m,t), \quad (19)$$

and $L_{ij}(s,h,l)$ satisfies the following system of equations:

$$L_{ij}(s,h,l) = 1_{\{l=h-s\}}\delta_{ij}[1-H_i(s,h)]+\sum_{k\in E}\sum_{m=s+1}^{h-l}q_{ik}(s,m)L_{kj}(m,h,l), \quad (20)$$

Applying these results we get:

$$E[\rho_1(\tau_s)v^{T\wedge\tau_s-s}1_{\{s<\tau_s\le T\}}] = \sum_{h=s+1}^{T} v^{h-s}\{R(s,h-1)-R(s,h)\}\times$$

$$\times\sum_{j\in Up} r_j \frac{\sum_{i\in E}\beta_i(s)\sum_{l=0}^{h-1-s}L_{ij}(s,h-1,l)\frac{q_{jD}(h-1-l,h)}{(1-H_j(h-1-l,h-1))}}{\sum_{i\in E}\beta_i(s)\sum_{k\in Up}\sum_{l=0}^{h-1-s}L_{ik}(s,h-1,l)\frac{q_{kD}(h-1-l,h)}{(1-H_k(h-1-l,h-1))}} \quad (21)$$

finally putting (21) in equation (14) we obtain the credit default swap spread:

$$U_1^*(s) = \frac{100(1-v)\Big\{\sum_{h=s+1}^{T}v^{h-s}\{R(s,h-1)-R(s,h)\}\times}{v[1-\sum_{h=s+1}^{T}v^{h-s}\{R(s,h-1)-R(s,h)\}-v^{T-s}R(s,T)]}\times$$

$$[1-\sum_{j\in Up} r_j \frac{\sum_{i\in E}\beta_i(s)\sum_{l=0}^{h-1-s}L_{ij}(s,h-1,l)\frac{q_{jD}(h-1-l,h)}{(1-H_j(h-1-l,h-1))}}{\sum_{i\in E}\beta_i(s)\sum_{k\in Up}\sum_{l=0}^{h-1-s}L_{ik}(s,h-1,l)\frac{q_{kD}(h-1-l,h)}{(1-H_k(h-1-l,h-1))}}]\Big\} \quad (22)$$

Note that we can assume a dependence of the recovery rate on the last n states visited by the process first of default time τ_s. We define the n-period stochastic recovery rate as

$$\rho_n(\tau_s) = \begin{cases} 1 & \text{if } s<T<\tau_s \\ \sum_{i=1}^{n}\alpha_{in}^{\tau_s}\rho_1(\tau_s-i+1) & \text{if } n+s\le\tau_s\le T \\ \rho_{\tau_s}(\tau_s) & \text{if } \tau_s<n+s \end{cases} \quad (23)$$

where $\rho_1(\tau_s-i+1) = r_j$ if $Z(\tau_s-i) = j$ and $Z(\tau_s) = D$, whereas $\alpha_{in}^{\tau_s}$ denote the proportion of the n period recovery rate with default time τ_s that depends on the one period recovery rate at time τ_s-i+1.

In such case to obtain the credit default swap spread we substitute the one period recovery rate in the equation (14) with the n-period one obtaining:

$$U_n^*(s) = \frac{(100[E[v^{T\wedge\tau_s-s}1_{\{s<\tau_s\le T\}}]-E[\rho_n(\tau_s)v^{T\wedge\tau_s-s}1_{\{s<\tau_s\le T\}}]])\times(1-v)}{v\times(1-E[v^{T\wedge\tau_s-s}])}. \quad (24)$$

If we choose $\underline{\alpha}$ such that $\alpha_{1n}^{\tau_s}=1$, $\alpha_{in}^{\tau_s}=0$ $\forall i \neq 1$ we obtain $U_n^*(s)=U_1^*(s)$.
All we need is to compute the unique new component $E[\rho_n(\tau_s)v^{T\wedge\tau_s-s}1_{\{s<\tau_s\leq T\}}]$.

$$E[\rho_n(\tau_s)v^{T\wedge\tau_s-s}1_{\{s<\tau_s\leq T\}}] = E[E[\rho_n(\tau_s)v^{T\wedge\tau_s-s}1_{\{s<\tau_s\leq T\}}|\tau_s]] \tag{25}$$

so we start computing the conditional expectation.

$$E[\rho_n(\tau_s)v^{(T\wedge\tau_s)-s}1_{\{s<\tau_s\leq T\}}|\tau_s] = \sum_{h=s+1}^{\infty} \rho_n(h)v^{(T\wedge h)-s}P[\tau_s = h]1_{\{h\leq T\}} =$$

$$\sum_{h=s+1}^{n+s-1} \rho_n(h)v^{h-s}P[\tau_s = h] + \sum_{h=n+s}^{T} \rho_n(h)v^{h-s}P[\tau_s = h] \tag{26}$$

now we apply the definition (23) and we get in

$$= \sum_{h=s+1}^{n+s-1} \rho_h(h)v^{h-s}P[\tau_s = h] + \sum_{h=n+s}^{T} \sum_{i=1}^{n} \alpha_{in}^h \rho_1(h-i+1)v^{h-s}P[\tau_s = h]$$

consequently $E[\rho_n(\tau_s)v^{(T\wedge\tau_s)-s}1_{\{s<\tau_s\leq T\}}] =$

$$\sum_{h=s+1}^{n+s-1} E[\rho_h(h)]v^{h-s}P[\tau_s = h] + \sum_{h=n+s}^{T} \sum_{i=1}^{n} \alpha_{in}^h E[\rho_1(h-i+1)]v^{h-s}P[\tau_s = h] \tag{27}$$

Now $E[\rho_1(h-i+1)] = \sum_{j\in Up} r_j P[Z(h-i) = j|Z(h) = D]$ and

$$E[\rho_h(h)] = \sum_{i=1}^{h} \alpha_{ih}^h E[\rho_1(h-i+1)] = \sum_{i=1}^{h} \alpha_{ih}^h \sum_{j\in Up} r_j P[Z(h-i) = j|Z(h) = D] \tag{28}$$

finally we obtain $E[\rho_n(\tau_s)v^{(T\wedge\tau_s)-s}1_{\{s<\tau_s\leq T\}}] =$

$$\sum_{h=s+1}^{n+s-1} \sum_{i=1}^{h} \alpha_{ih}^h \sum_{j\in Up} r_j P[Z(h-i) = j|Z(h) = D]v^{h-s}P[\tau_s = h]+$$

$$+ \sum_{h=s+n}^{T} \sum_{i=1}^{n} \alpha_{in}^h \sum_{j\in Up} r_j P[Z(h-i) = j|Z(h) = D]v^{h-s}P[\tau_s = h] \tag{29}$$

The probabilities $P[Z(h-i) = j|Z(h) = D]$ can be evaluated by the Bayes formula, in fact $\forall h, i \in \mathbb{N}$ such that $h - i \geq s$

$$P[Z(h-i) = j|Z(h) = D] =$$

$$\frac{\sum_{l=0}^{h-i-s} P[Z(h)=D|Z(h-i)=j, B(h-i)=l]P[Z(h-i)=j, B(h-i)=l]}{\sum_{k\in Up}\sum_{l=0}^{h-i-s}P[Z(h)=D|Z(h-i)=k, B(h-i)=l]P[Z(h-i)=k, B(h-i)=l]} =$$

$$\frac{\sum_{i \in E} \beta_i(s) \sum_{l=0}^{h-i-s} L_{ij}(s,h-i,l) \Delta_{jD}(h-i,l,h)}{\sum_{i \in E} \beta_i(s) \sum_{k \in Up} \sum_{l=0}^{h-i-s} L_{ik}(s,h-i,l) \Delta_{kD}(h-i,l,h)}. \tag{30}$$

At this point we substitute (30) in (29) and the obtained (29) in (27). Finally we insert (27) together with (7) in (24) and we obtain $U_n^*(s)$.

We conclude noting that the evaluation procedure in case of random recovery rate doesn't present problems, in fact we have only to change the outflow's definition that in this case is

$$O(s,t) = 100(1 - \rho_n(\tau_s)) v^{(T \wedge \tau_s)-t} 1_{\{t < \tau_s \le T\}}$$

which expectation can be evaluated using computations similar as those used to determine the credit default swap spread corresponding to a random recovery rate.

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