Managing Value-at-Risk for a bond using bond put options

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Abstract. This paper studies a strategy that minimizes the Value-at-Risk (VaR) of a position in a zero coupon bond by buying a percentage of a put option, subject to a fixed budget available for hedging. We elaborate a formula for determining the optimal strike price for this put option in case of a Vasicek stochastic interest rate model. We demonstrate the relevance of searching the optimal strike price, since moving away from the optimum implies a loss, either due to an increased VaR or due to an increased hedging expenditure. In this way, we extend the results of [Ahn *et al.*, 1999], who minimize VaR for a position in a share. In addition, we look at the alternative risk measure Tail Value-at-Risk.

Keywords: Value-at-Risk, bond hedging, Vasicek interest rate model.

1 Introduction

Many financial institutions and non-financial firms nowadays publicly report Value-at-Risk (VaR), a risk measure for potential losses. Internal uses of VaR and other sophisticated risk measures are on the rise in many financial institutions, where, for example, a bank risk committee may set VaR limits, both amounts and probabilities, for trading operations and fund management. At the industrial level, supervisors use VaR as a standard summary of market risk exposure. An advantage of the VaR measure, following from extreme value theory, is that it can be computed without full knowledge of the return distribution. Semi-parametric or fully non-parametric estimation methods are available for downside risk estimation. Furthermore, at a sufficiently low confidence level the VaR measure explicitly focuses risk managers and regulators attention on infrequent but potentially catastrophic extreme losses.

Value-at-Risk (VaR) has become the standard criterion for assessing risk in the financial industry. Given the widespread use of VaR, it becomes increasingly important to study the effects of options on the VaR-based risk management.

The starting point of our analysis is the classical hedging example, where an institution has an exposure to the price risk of an underlying asset. This may be currency exchange rates in the case of a multinational corporation, oil prices in the case of an energy provider, gold prices in the case of a mining company, etc. The corporation chooses VaR as its measure of market risk. Faced with the unhedged VaR of the position, we assume that the institution chooses to use options and in particular put options to hedge a long position in the underlying.

[Ahn *et al.*, 1999] consider the problem of hedging the Value-at-Risk of a position in a single share by investing a fixed amount C in a put option.
The principal purpose of our study is to extend these results to the situation of a bond. We consider the well-known continuous-time stochastic interest rate model of [Vasicek, 1977] to investigate the optimal speculative and hedging strategy based on this framework by minimizing the Value-at-Risk of the bond, subject to the fixed amount C which is spent on put options. In addition, we consider an alternative risk measure Tail Value-at-Risk (TVaR), for which we solve the minimization problem and obtain the optimal hedging policy. In further versions, we will elaborate this part more deeply.

The discussion is divided as follows: Section 2 presents the general risk management model, introduces the Vasicek model and considers hedging with bond put options. Afterwards, Section 3 discusses the optimal hedging policy for VaR, considers the closely related risk measure TVaR and introduces comparative statics. Section 4 consists of a numerical illustration. Finally, Section 5 summarizes the paper, concludes and introduces further research possibilities.

2 The mathematical framework

Consider a portfolio with value W_t at time t. The Value-at-Risk of this portfolio is defined as the $(1-\alpha)$-quantile of the loss distribution depending on a time interval with length T. Common time periods that are taken into consideration are $T = 1$, 10, 20 days. A formal definition for the $\mathrm{VaR}_{\alpha,T}$ is

$$\Pr(W_0 - W_T^d \geq \mathrm{VaR}_{\alpha,T}) = \alpha,$$

with W_T^d the value of the portfolio at time T, discounted back until time zero by means of a zero coupon with maturity T.
In other words $\mathrm{VaR}_{\alpha,T}$ is the loss of the worst case scenario on the investment at a $1-\alpha$ confidence level during the period $[0, T]$. It is possible to define the $\mathrm{VaR}_{\alpha,T}$ in a more general way

$$\mathrm{VaR}_{\alpha,T} = \inf\left\{Y \mid \Pr(W_0 - W_T^d \geq Y) < \alpha\right\}.$$

In this study, we focus on the hedging problem of a zero-coupon bond. Therefore, we need to define a process that describes the evolution of the instantaneous interest rate, and enables us to value the zero-coupon bond. As term structure model, we consider the Vasicek model, which is a typical example of an affine term structure model.

2.1 The Vasicek model

[Vasicek, 1977] assumes that the instantaneous interest rate follows a mean reverting process also known as an Ornstein-Uhlenbeck process:

$$dr(t) = \kappa(\theta - r(t))dt + \sigma dZ(t) \tag{1}$$

for a standard Brownian motion $Z(t)$ under the risk-neutral measure Q, and with constants κ, θ and σ. The parameter κ controls the mean-reversion speed, θ is the long-term average level of the spot interest rate around which $r(t)$ moves, and σ is the volatility measure. The reason of the Vasicek model's popularity is its analytical and mathematical tractability. An often cited critique is that applying the model sometimes results in a negative interest rate.

It can be shown that the expectation and variance of the stochastic variable $r(t)$ are:

$$E\left[r(t)\right] = m = \theta + (r(0) - \theta)e^{-\kappa t} \tag{2}$$

$$\mathrm{Var}\left[r(t)\right] = s^2 = \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa t}). \tag{3}$$

Based on these results, Vasicek develops an analytical expression for the price of a zero-coupon bond with maturity date S

$$Y(t,S) = \exp[A(t,S) - B(t,S)r(t)], \tag{4}$$

where

$$B(t,S) = \frac{1 - e^{-\kappa(S-t)}}{\kappa}, \tag{5}$$

$$A(t,S) = (B(t,S) - (S-t))(\theta - \frac{\sigma^2}{2\kappa^2}) - \frac{\sigma^2}{4\kappa}B(t,S)^2. \tag{6}$$

Since $A(t,S)$ and $B(t,S)$ are independent of $r(t)$, the distribution of a bond price at any given time must be lognormal with parameters Π and Σ^2:

$$\Pi(t,S) = A(t,S) - B(t,S)m, \qquad \Sigma(t,S)^2 = B(t,S)^2 s^2, \tag{7}$$

with m and s^2 given by (2) and (3).

From the formulae (4)-(7), we can see that for $S \geq T$ the present value of the loss of the (unhedged) portfolio can be expressed as function of z

$$L_0 = W_0 - W_T^d = Y(0,S) - Y(0,T)e^{\Pi(T,S)+\Sigma(T,S)z} = f(z) \tag{8}$$

where f is a strictly decreasing function and z is a stochastic variable with a standard normal distribution. Therefore, the $\mathrm{VaR}_{\alpha,T}$ of such a portfolio is determined by the formula

$$\mathrm{VaR}_{\alpha,T} = f(c(\alpha)), \tag{9}$$

where $c(\alpha)$ is the cut off point for the standard normal distribution at a certain percent level i.e. $\Pr(z \leq c(\alpha)) = \alpha$.

Since the distribution of the unhedged position in the zero-coupon bond is lognormal in the Vasicek model, from the formulae (8)-(9) we observe that the Value-at-Risk measure for the zero-coupon bond can be expressed as

$$\mathrm{VaR}_{\alpha,T} = Y(0,S) - Y(0,T)e^{\theta_B(\alpha)},$$

where

$$\theta_B(\alpha) = \Pi(T,S) + \Sigma(T,S)c(\alpha) \tag{10}$$

and $c(\cdot)$ is the percentile of the standard normal distribution.

2.2 Put options and hedging

We recall from [Ahn *et al.*, 1999] the classical hedging example, where an institution has an exposure to the price risk of an underlying asset S_T. The hedged future value of this portfolio at time T is given by

$$H_T = \max(hX + (1-h)S_T, S_T), \tag{11}$$

where $0 \leq h \leq 1$, represents the hedge ratio, that is, the percentage of put option P used in the hedge and X is the strike price of the option.

In our setup, the underlying security is a bond and the hedging tool is a bond put option, the price of which will be worked out hereafter.
We recall that the price of a European call option with the zero-coupon bond which matures at time S as the underlying security and with strike price X and exercise date T (with $T \leq S$) is at date t given by:

$$C(t,T,X) = Y(t,S)\Phi(d_1) - XY(t,T)\Phi(d_2), \tag{12}$$

where

$$d_1 = \frac{1}{\sigma_p}\log\left(\frac{Y(t,S)}{XY(t,T)}\right) + \frac{\sigma_p}{2}, \qquad d_2 = d_1 - \sigma_p,$$

$$\sigma_p = \frac{\sigma}{\kappa}(1 - e^{-\kappa(S-T)})\sqrt{\frac{1-e^{-2\kappa(T-t)}}{2\kappa}},$$

and $\Phi(z)$ is the cumulative distribution function of a standard normal random variable. The Put-Call parity model is designed to determine the value of a put option from a corresponding call option and provides in this case the following European put option price corresponding to (12):

$$P(t,T,X) = -Y(t,S)\Phi(-d_1) + XY(t,T)\Phi(-d_2). \tag{13}$$

3 The bond hedging problem

3.1 VaR minimization

Analogously to [Ahn *et al.*, 1999], we assume that we have one bond and we use only a percentage of a put option on the bond to hedge. We will find the optimal strike price which minimizes VaR for a given hedging cost.
Indeed, let us assume that the institution has an exposure to a bond, $Y(0,S)$, which matures at time S, and that the company has decided to hedge the bond value by using a percentage of one put option $P(0,T,X)$ with strike price X and exercise date T (with $T \leq S$). Then we can look at the future value of the hedged portfolio (which is composed of the bond Y and the put option $P(0,T,X)$) at time T as a function, analogously to (11), of the form

$$H_T = \max(hX + (1-h)Y(T,S), Y(T,S)).$$

If the put option finishes in-the-money (a case which is of interest to us), then the discounted value of the future value of the portfolio is

$$H_T^d = ((1-h)Y(T,S) + hX)Y(0,T).$$

Taking into account the cost of setting up our hedged portfolio, which is given by the sum of the bond price $Y(0,S)$ and the cost C of the position in the put option, we get for the present value of the loss

$$L_0 = Y(0,S) + C - ((1-h)Y(T,S) + hX)Y(0,T),$$

and this under the assumption that the put option finishes in-the-money.
We recall that $Y(T,S)$ has a lognormal distribution with parameters Π and Σ^2, given by (7). Therefore the loss function equals

$$Y(0,S) + C - ((1-h)e^{\Pi(T,S)+\Sigma(T,S)z} + hX)Y(0,T),$$

where z again denotes a stochastic variable with a standard normal distribution. The Value-at-Risk at an α percent level of a position $H = \{Y, h, P\}$ consisting of a bond Y and h put options P (which are assumed to be in-the-money) with a strike price X and an expiry date T is equal to

$$\mathrm{VaR}_{\alpha,T}(L_0) = Y(0,S) + C - ((1-h)e^{\theta_B(\alpha)} + hX)Y(0,T), \qquad (14)$$

where we recall that $\theta_B(\alpha) = \Pi + \Sigma c(\alpha)$ and $c(\alpha)$ is the percentile of the standard normal distribution.

Similar to the Ahn et al. problem, we would like to minimize the risk of the future value of the hedged bond H_T, given a maximum hedging expenditure C. More precisely,

$$\min_X Y(0,S) + C - ((1-h)e^{\theta_B(\alpha)} + hX)Y(0,T)$$

subject to the restrictions $C = hP(0,T,X)$ and $h \in (0,1)$.
Solving this constrained optimization problem, we find that the optimal strike price X^* satisfies the following equation

$$P(0,T,X) - (X^* - e^{\theta_B(\alpha)})\frac{\partial P(0,T,X)}{\partial X} = 0.$$

or equivalently, when taking (13) into account,

$$e^{\theta_B(\alpha)} = \frac{Y(0,S)\Phi(-d_1)}{Y(0,T)\Phi(-d_2)}. \tag{15}$$

We note that the optimal strike price is independent of the hedging cost.

3.2 Tail VaR minimization

In this section, we introduce the concept of Tail Value-at-Risk, TVaR, also known as mean excess loss, mean shortfall or Conditional VaR. We further demonstrate the ease of extending our analysis to this alternative risk measure.

A drawback of the traditional Value-at-Risk measure is that it does not care about the tail behaviour of the losses. In other words, by focusing on the VaR at, let's say a 5% level, we ignore the potential severity of the losses below that 5% threshold. In other words, we have no information on how bad things can become in a real stress situation. Therefore, the important question of 'how bad is bad' is left unanswered. TVaR is trying to capture this problem by considering the possible losses, once the VaR threshold is crossed.

Formally,

$$\mathrm{TVaR}_{\alpha,T} = \frac{1}{\alpha}\int_{1-\alpha}^{1} \mathrm{VaR}_{1-\beta,T}\, d\beta.$$

This formula boils down to taking the arithmetic average of the quantiles of our loss, from $1-\alpha$ to 1 on, where we recall that $\mathrm{VaR}_{\alpha,T}$ stands for the quantile at the level $1-\alpha$.

If the cumulative distribution function of the loss is continuous, which is the case in our problem, TVaR is equal to the Conditional Tail Expectation (CTE) which for the loss L_0 is calculated as:

$$\mathrm{CTE}_{\alpha,T}(L_0) = E[L_0 \mid L_0 > \mathrm{VaR}_{\alpha,T}(L_0)].$$

A closely related risk measure concerns Expected Shortfall (ESF). It is defined as:

$$\mathrm{ESF}(L_0) = E\left[(L_0 - \mathrm{VaR}_{\alpha,T}(L_0))_+\right].$$

In order to determine $\mathrm{TVaR}_{\alpha,T}(L_0)$, we can make use of the following equality:

$$\begin{aligned}\mathrm{TVaR}_{\alpha,T}(L_0) &= \mathrm{VaR}_{\alpha,T}(L_0) + \frac{1}{\alpha}\mathrm{ESF}(L_0)\\ &= \mathrm{VaR}_{\alpha,T}(L_0) + \frac{1}{\alpha}E\left[(L_0 - \mathrm{VaR}_{\alpha,T}(L_0))_+\right].\end{aligned}$$

This formula already makes clear that $\mathrm{TVaR}_{\alpha,T}(L_0)$ will always be larger than $\mathrm{VaR}_{\alpha,T}(L_0)$.
In our case, the loss has a lognormal distribution, because of the lognormality of our bond prices. This allows us to write the ESF as

$$\mathrm{ESF}(L_0) = (1-h)Y(0,T)e^{\Pi(T,S)}\Big[\alpha e^{\Sigma(T,S)c(\alpha)} - e^{\frac{1}{2}\Sigma^2(T,S)}\Phi(c(\alpha) - \Sigma(T,S))\Big].$$

This reduces our $\mathrm{TVaR}_{\alpha,T}(L_0)$ to:

$$\begin{aligned}\mathrm{TVaR}_{\alpha,T}(L_0) &= Y(0,S) + C - hXY(0,T)\\ &\quad -\frac{1}{\alpha}(1-h)e^{\Pi(T,S)+\frac{1}{2}\Sigma^2(T,S)}\Phi(c(\alpha) - \Sigma(T,S))Y(0,T).\end{aligned}$$

We again seek to minimize this TVaR, in order to minimize potential losses. The procedure for minimizing this TVaR is analogue to the VaR minimization procedure. The resulting optimal strike price can thus be determined from the formula below:

$$\frac{1}{\alpha}e^{\Pi(T,S)+\frac{1}{2}\Sigma^2(T,S)}\Phi(c(\alpha) - \Sigma(T,S)) = \frac{Y(0,S)\Phi(-d_1)}{Y(0,T)\Phi(-d_2)}.$$

3.3 Comparative statics

We examine how changes in the parameters of the Vasicek model influence the optimal strike price, by means of the derivatives of the optimal strike price with respect to these parameters.

For both $\mathrm{VaR}_{\alpha,T}$ and $\mathrm{TVaR}_{\alpha,T}$, the optimal strike price is implicitly defined by

$$F(X,\beta) = \mathrm{FAC}\cdot Y(0,T)\Phi(-d_2) - Y(0,S)\Phi(-d_1) = 0,$$

with β the vector including the Vasicek parameters, that is θ, κ and the volatility σ, see Section 2.1, and with FAC representing $e^{\theta_B(\alpha)}$ in the case of $\mathrm{VaR}_{\alpha,T}$ and $\frac{1}{\alpha}e^{\Pi(T,S)+\frac{1}{2}\Sigma^2(T,S)}\Phi(c(\alpha) - \Sigma(T,S))$ in the case of $\mathrm{TVaR}_{\alpha,T}$.
Taking into account the implicit function theorem, we obtain the required derivatives as follows:

$$\frac{\partial F}{\partial X}dX + \frac{\partial F}{\partial \beta}d\beta = 0 \Longleftrightarrow \frac{dX}{d\beta} = -\frac{\frac{\partial F}{\partial \beta}}{\frac{\partial F}{\partial X}}. \tag{16}$$

The denominator of (16) is equal for the different derivatives, and is given by

$$\frac{\partial F}{\partial X} = \frac{\mathrm{FAC} \cdot Y(0,T)\varphi(d_2) - Y(0,S)\varphi(d_1)}{X\sigma_p}, \tag{17}$$

with φ being the density function of a standard normal random variable, while the numerator of (16) can be obtained by applying the following formula,

$$\begin{aligned} \frac{\partial F}{\partial \beta} = \frac{\partial \mathrm{FAC}}{\partial \beta} Y(0,T)\Phi(-d_2) + \mathrm{FAC} \cdot \frac{\partial Y(0,T)}{\partial \beta}\Phi(-d_2) \\ -\mathrm{FAC} \cdot Y(0,T)\varphi(d_2)\frac{\partial d_2}{\partial \beta} - \frac{\partial Y(0,S)}{\partial \beta}\Phi(-d_1) + Y(0,S)\varphi(d_1)\frac{\partial d_1}{\partial \beta}. \end{aligned} \tag{18}$$

These derivatives are rather involved and do not lead to a straightforward interpretation of their sign and magnitude. Therefore, we will describe the derivatives in the next paragraph using a numerical illustration.

Further relevant derivatives are $\frac{dX}{dS}$ and $\frac{dX}{dT}$ to study the response of the optimal strike price to a change in the maturity of both the underlying bond and the maturity of the bond option used to hedge the exposure. They follow from formulae (16)-(18), after having replaced β by S and T respectively, and taking into account the simplification due to the fact that $Y(0,T)$ is independent of S, and $Y(0,S)$ is independent of T. Again, we leave the interpretation of these derivatives to the next section.

A last derivative of interest is the one with respect to α, formally $\frac{dX}{d\alpha}$:

$$\frac{dX}{d\alpha} = -\frac{1}{\frac{\partial F}{\partial X}} \cdot \frac{\partial \mathrm{FAC}}{\partial \alpha} Y(0,T)\Phi(-d_2),$$

where $\frac{\partial \mathrm{FAC}}{\partial \alpha}$ is respectively given by

$$\frac{e^{\theta_B(\alpha)}\Sigma(T,S)}{\varphi(c(\alpha))} \qquad \text{(VaR)}$$

$$\frac{e^{\Pi(T,S)+\frac{1}{2}\Sigma^2(T,S)}}{\alpha^2}\left[\frac{\alpha\varphi(c(\alpha)-\Sigma(T,S))}{\varphi(c(\alpha))} - \Phi(c(\alpha)-\Sigma(T,S))\right] \quad \text{(TVaR).}$$

4 Numerical results

We illustrate the usefulness of the above results for the VaR case (TVaR case is ongoing research). In order to provide a credible numerical illustration, we take the parameter estimates for the Vasicek model from [Chan *et al.*, 1992], who compare a variety of continuous-time models of the short term interest rate with respect to their ability to fit the U.S. Treasury bill yield. This results in the following parameter values: $\sigma = 0.02$, $\theta = 0.0866$, $\kappa = 0.1779$, $r(0) = 0.06715$. Next, we should consider the budget the financial institution is willing to spend on the hedging. Standardising the nominal value of the bond at issuance to 1, we start with a hedging budget of 0.05, so $C = 0.05$. We also assume the bank is considering the VaR at the five percent level, meaning that $\alpha = 5\%$.

We considered two situations, one in which the bank wishes to hedge a bond with a maturity of one year ($S = 1$), and one for a bond with a maturity of ten year ($S = 10$).
We observe that our strategy is successful in decreasing the risk, while, since we use options, still providing us with upward potential. In the one year bond case, the mean reduction in VaR (calculated as the difference between the VaR of the hedged position and the VaR of the unhedged position, divided by VaR of the unhedged position) over the holding period amounts to 6.25%. The maximum reduction is 26.23%, whereas the lowest reduction is 3.25%. In the ten year bond case, the mean VaR reduction over the holding period is 5.36%. The maximum reduction that can be achieved amounts to 26.15%. The minimum reduction is 2.59%.

As already mentioned above, we are also interested in the effect of changes in the parameter estimates of the Vasicek model on the optimal strike price. We examine these effects using the first example, in which the bond matures in one year. An increase in one of these parameters always leads to a lower optimal strike price. The influence of a 1% increase in κ only marginally effects the strike price. Changes in θ also have a moderate impact on the optimal strike. The most influential parameter of the Vasicek model undoubtedly is the volatility. Whereas for κ and θ the impact constantly decreases as the holding period comes closer to the maturity of the bond, we find a non-monotonic relationship between the derivative (with respect to the volatility) and the difference between the holding period T and the maturity S of the bond.

Increasing the maturity of the bond decreases the strike price, while increasing the holding period (meaning that the holding period moves closer to the maturity of the bond) increases the strike price. Reducing the certainty with which a bank wishes to know the value it can lose, or in other words, increasing α leads to a increased strike price. This increase again depends on the holding period in a non monotonic way.

5 Conclusion

In this paper, we studied the optimal risk control for one bond using a percentage of a put option by means of Value-at-Risk and Tail Value-at-Risk, widespread concepts in the financial world. The interest model we use for valuation, is the Vasicek model. The optimal strategy corresponds to buying a put option with optimal strike price in order to have a minimal VaR or TVaR given a fixed hedging cost. We did not obtain an explicit result, but numerical methods can be easily implemented to solve for the optimal strategy. For the VaR case, we demonstrate the relevance of searching for this optimal strike price, since moving away from this optimum implies a loss, either because of an increased VaR, or an increased hedging expenditure. For TVaR, the numerical illustration is part of ongoing research.

Further analysis is oriented in a number of directions. First of all, we plan to examine the implications of assuming a different interest rate model e.g. Hull-White. We will further turn to a deeper study of the effects in the optimal hedging policy of using either VaR or TVaR.

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