Brownian Laplace motion and its use in financial modelling

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Abstract. A new Lévy motion with both continuous (Brownian) and discontinuous (Laplace motion) components is introduced. The increments of the process follow a *generalized normal Laplace* (GNL) distribution, which exhibits positive kurtosis and can be either symmetrical or skewed. The degree of kurtosis in the increments increases as the length of the increment decreases. This and other properties of Brownian-Laplace motion refelect those of observed time series of logarithmic stock-price returns and thus render it a good model for fitting to financial data and for the calculation of the theoretical value of financial derivatives. A formuala for the value of European call options based on Brownian-Laplace motion is given.
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1 Introduction.

The Black-Scholes theory of option pricing was originally based on the assumption that asset prices follow geometric Brownian motion (GBM). For such a process the logarithmic returns ($\log(P_{t+1}/P_t)$ on the price P_t are independent identically distributed (iid) normal random variables. However it has been recognized for some time now that the logarithmic returns do not behave quite like this, particulary over short intervals. Empirical distributions of the logarithmic returns in high-frequency data usually exhibit excess kurtosis with more probability mass near the origin and in the tails and less in the flanks than would occur for normally distributed data. Furthermore the degree of excess kurtosis is known to increase as the sampling interval decreases (see *e.g.* [Rydberg, 2000]). In addition skewness can sometimes be present. To accomodate for these facts new models for price movement based on Lévy motion have been developed (see *e.g.* [Schoutens, 2003]). For any infinitely divisible distribution a Lévy process can be contructed whose increments follow the given distribution. Thus in modelling financial data one needs to find an infinitely divisible distribution which fits well to observed logarithmic returns. A number of such distributions have been suggested including the gamma, inverse Gaussian, Laplace (or variance gamma), Meixner and generalized hyperbolic distributions (see [Schoutens, 2003] for details and references).

In this paper a new infinitely divisible distribution – the *generalized normal Laplace* (or GNL) distribution – which exhibits the properties seen in observed logarithmic returns, is introduced. This distribution arises as the sum of independent normal and generalized Laplace [Kotz *et al.*, 2001] random variables[11]. A Lévy process based on the generalized Laplace (variance-gamma) distribution alone has no Brownian component, only linear deterministic and pure jump components *i.e.* its Lévy-Khintchine triplet is of the form $(\gamma, 0, \nu(dx))$ (see [Schoutens, 2003]). The new distribution of this paper in effect adds a Brownian component to this motion, leading to what will be called *Brownian-Laplace motion*[22].

In the following section the generalized normal Laplace (GNL) distribution is defined and some properties given. Brownian-Laplace motion is then defined as a Lévy process whose increments follow the GNL distribution. In Sec. 3 a pricing formula is developed for European call options on a stock whose logarithmic price follows Brownian-Laplace motion.

2 The generalized normal Laplace (GNL) distribution.

The *generalized normal Laplace* (GNL) distribution is defined as that of a random variable Y with characteristic function

$$\phi(s) = \left[\frac{\alpha\beta\exp(\mu is - \sigma^2 s^2/2)}{(\alpha - is)(\beta + is)}\right]^\rho \tag{1}$$

where α, β, ρ and σ are positive parameters and $-\infty < \mu < \infty$. We shall write

$$Y \mathrm{sim} \mathrm{GNL}(\mu, \sigma^2, \alpha, \beta, \rho)$$

to indicate that the random variable Y follows such a distribution.
Since the characteristic function (1) can be written

$$\exp(\rho\mu is - \rho\sigma^2 s^2/2)\left[\frac{\alpha}{\alpha - is}\right]^\rho\left[\frac{\beta}{\beta + is}\right]^\rho$$

it follows that Y can be represented as

$$Y \stackrel{d}{=} \rho\mu + \sigma\sqrt{\rho}Z + \frac{1}{\alpha}G_1 - \frac{1}{\beta}G_2 \tag{2}$$

[1] 1. The generalized asymmmetric Laplace distribution is better known as the variance-gamma distribution in the finance literature. It is also known as the Bessel K-function distribution (see [Kotz *et al.*, 2001], for a discussion of the terminology and history of this distribution).

[2] 2. An alternative name, which invokes two of the greatest names in the history of mathematics, would be *Gaussian-Laplace motion*

where Z, G_1 and G_2 are independent with Zsim N(0,1) and G_1, G_2 gamma random variables with scale parameter 1 and shape parameter ρ, *i.e.* with probability density function (pdf)

$$g(x) = \frac{1}{\Gamma(\rho)} x^{\rho-1} e^{-x}.$$

This representation provides a straightforward way to generate pseudo-random deviates following a GNL distribution. Note from (1) it is easily established that the GNL is infinitely divisible. In fact the n-fold convolution of a GNL random variable also follows a GNL distribution.

The mean and variance of the GNL$(\mu, \sigma^2, \alpha, \beta, \rho)$ distribution are

$$\mathrm{E}(Y) = \rho \left(\mu + \frac{1}{\alpha} - \frac{1}{\beta} \right); \qquad \mathrm{var}(Y) = \rho \left(\sigma^2 + \frac{1}{\alpha^2} + \frac{1}{\beta^2} \right)$$

while the higher order cumulants are (for $r > 2$)

$$\kappa_r = \rho(r-1)! \left(\frac{1}{\alpha^r} + (-1)^r \frac{1}{\beta^r} \right).$$

The parameters μ and σ^2 influence the central location and spread of the distribution, while α and β affect the lengths of the tails. *Ceteris paribus* decreasing α (or β) puts more weight into the upper (or lower) tail. The tail behaviour of the GNL distribution can be determined from the nature of the poles of its characteristic (or moment generating) function (see *e.g.* [Doetsch, 1970]). In the tails the generalized Laplace component of the GNL dominates - precisely $f(y)\mathrm{sim} c_1 y^{\rho-1} e^{-\alpha y}$ $(y \to \infty)$ and $f(y)\mathrm{sim} c_2 (-y)^{\rho-1} e^{\beta y}$ $(y \to -\infty)$, (where c_1 and c_2 are constants). Thus for $\rho < 1$, both tails are fatter than exponential; for $\rho = 1$ they are exactly exponential and for $\rho > 1$ they are less fat than exponential.

The parameter ρ affects all moments. However the coefficients of skewness $(\gamma_1 = \kappa_3/\kappa_2^{3/2})$ and of kurtosis $(\gamma_2 = \kappa_4/\kappa_2^2)$ both decrease with increasing ρ (and converge to zero as $\rho \to \infty$) with the shape of the distribution becoming more normal with increasing ρ, (exemplifying the central limit effect since the sum of n iid GNL$(\mu, \sigma^2, \alpha, \beta, \rho)$ random variables has a GNL$(\mu, \sigma^2, \alpha, \beta, n\rho)$ distribution).

When $\alpha = \beta$ the distribution is symmetric. In the limiting case $\alpha = \beta = \infty$ the GNL reduces to a normal distribution.

3 A Lévy process based on the GNL distribution - Brownian-Lapace motion.

Consider now a Lévy process $\{X_t\}_{t \geq 0}$, say for which the increments $X_{t+\tau} - X_\tau$ have characteristic function $(\phi(s))^t$ where ϕ is the characteristic function (1) of the GNL($\mu, \sigma^2, \alpha, \beta, \rho$) distribution (such a construction is always possible for an infinitely divisible distribution - see [Schoutens, 2003]. It is not difficult to show that the Lévy-Khintchine triplet for this process is $(\rho\mu, \rho\sigma^2, \Lambda)$ where Λ is the Lévy measure of asymmetric Laplace motion (see Kotz *et al.*, 2001, p.196). Laplace motion has an infinite number of jumps in any finite time interval (a pure jump process). The extension considered here adds a continuous Brownian component to Laplace motion leading to the name *Brownian-Laplace motion.*

The increments $X_{t+\tau} - X_\tau$ of this process will follow a GNL($\mu, \sigma^2, \alpha, \beta, \rho t$) distribution and will have fatter tails than the normal − indeed fatter than exponential for $\rho t < 1$. However as t increases the kurtosis of the distribution drops, and approaches zero as $t \to \infty$. Exactly this sort of behaviour has been observed in various studies on high-frequency financial data (*e.g.* [Rydberg, 2000]) - very little kurtosis in the distribution of logarithmic returns over long intervals but increasingly fat tails as the reporting interval is shortened. Thus Brownian-Laplace motion seems to provide a good model for the movement of logarithmic prices.

3.1 Option pricing for assets with logarithmic prices following Brownian-Laplace motion.

We consider an asset whose price S_t is given by

$$S_t = S_0 \exp(X_t)$$

where $\{X_t\}_{t\geq 0}$ is a Brownian-Laplace motion with $X_0 = 0$ and parameters $\mu, \sigma^2, \alpha, \beta, \rho$. We wish to determine the risk-neutral valuation of a European call option on the asset with strike price K at time T and risk-free interest rate r.

It can be shown using the Esscher equivalent martingale measure (see *e.g.* [Schoutens, 2003]) that the option value can be expressed in a form similar to that of the Black-Scholes formula. Precisely

$$OV = S_0 \int_\gamma^\infty d_{GNL}^{*T}(x; \theta + 1)dx - e^{-rT} K \int_\gamma^\infty d_{GNL}^{*T}(x; \theta)dx \tag{3}$$

where $\gamma = \log(K/S_0)$ and

$$d_{GNL}^{*T}(x; \theta) = \frac{e^{\theta x} d_{GNL}^{*T}(x)}{\int_{-\infty}^{\infty} e^{\theta y} d_{GNL}^{*T}(y)dy} \tag{4}$$

is the pdf of X_T under the risk-neutral measure. Here d_{GNL}^{*T} is the pdf of the T-fold convolution of the generalized normal-Laplace, GNL($\mu, \sigma^2, \alpha, \beta, \rho$), distribution and θ is the unique solution to the following equation involving the moment generating function (mgf) $M(s) = \phi(-is)$

$$\log M(\theta + 1) - \log M(\theta) = r. \tag{5}$$

The T-fold convolution of GNL$(\mu, \sigma^2, \alpha, \beta, \rho)$ is GNL$(\mu, \sigma^2, \alpha, \beta, \rho T)$ and so its moment generating function is (from (1))

$$M(s) = \left[\frac{\alpha\beta \exp(\mu s + \sigma^2 s^2/2)}{(\alpha - s)(\beta + s)} \right]^{\rho T}.$$

This provides the denominator of the expression (4) for the risk-neutral pdf.

Now let

$$I_\theta = \int_\gamma^\infty d_{GNL}^{*T}(x;\theta)dx = \frac{1}{[M(\theta)]^T} \int_\gamma^\infty e^{\theta x} d_{GNL}^{*T}(x) \tag{6}$$

so that

$$OV = S_0 I_{\theta+1} - e^{-rT} K I_\theta.$$

Thus to evaluate the option value we need only evaluate the integral in (6). This can be done using the representation (2) of a GNL random variable as the sum of normal and positive and negative gamma components. Precisely the integral can be written

$$\int_0^\infty g(u;\alpha) \int_0^\infty g(v;\beta) \int_\gamma^\infty e^{\theta x} \frac{1}{\sigma\sqrt{\rho T}} \phi\left(\frac{x - u + v - \mu\rho T}{\sigma\sqrt{\rho T}} \right) dxdvdu \tag{7}$$

where

$$g(x;a) = \frac{a^{\rho T}}{\Gamma(\rho T)} x^{\rho T - 1} e^{-ax}$$

is the pdf of a gamma random variable with scale parameter a and shape parameter ρT; and ϕ is the pdf of a standard normal deviate. After completing the square in x and evaluating the x integral in terms of Φ^c, the complementary cdf of a standard normal, (6) can be expressed

$$I_\theta = \int_0^\infty g(u;\alpha - \theta) \int_0^\infty g(v;\beta + \theta) \Phi^c \left(\frac{\gamma - u + v - \mu\rho T - \theta\sigma^2\rho T}{\sigma\sqrt{\rho T}} \right) dvdu. \tag{8}$$

For given parameter values the double integral (8) can be evaluated numerically quite quickly and thence the option value computed. For an example see [Reed, 2005].

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