An Application Of The Stochastic Mcshane'S Equations In Financial Modelling

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Abstract. We prove a result for the solvability of linear forward-backward stochastic differential equations of McShane type. The motivation for the study is a similar Black-Scholes type model in mathematical finance.
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1 Introduction

One of the most remarkable applications of stocahstic analysis is in mathematical finance. In particular, the Black-Scholes model enjoys great popularity (see, for example, [Musiela and Rutkowski, 1997]). Recently (see [Ma and Yong, 1999]), this model was derived by means of the theory of forward-backward stochastic differential equations of Itô type a setting appropriate for the case in which the filtration of the undelying probability space is given by Brownian motion. It appears that for a filtration induced by a finite variation process with a.s. continous sample paths, the Itô type stochastic integral is no longer appropriate. In this case one can use a McShane type integral (introduced by McShane in [McShane, 1969], [McShane, 1974] and further developed by Srinivasan in [Srinivasan, 1978] and, from a different point of wiev, by Protter in [Protter, 1992]; so a stochastic calculus could be called "unified calculus" since it includes ordinary calculus as a special case and also Itô Calculus) to construct a suitable model. This leads us to forward-backward stochastic differential equations of McShane type. Despite many investigation related to McShane type stochastic differential equations (see [Angulo Ibanez and Gutierrez Jaimez, 1988], [Constantin, 1998], [Ladde and Seikkala, 1986], [McShane, 1974] for theoretical approaches and [Srinivasan, 1978], [Srinivasan, 1984], [Hangii, 1980] for applications of McShane stochastic calculus to problems in physics) a study of forward-backward stochastic differential equations of McShane type has not been undertaken, to the best of our knowledge.

Stochastic calculus appears to be one of the natural tools for the study of models of those phenomena having some non-deterministic elements. For example, in the description of brownian motion the stochastic nature is adequately described by a linear differential equation with a random forcing term which is identified as a white noise process or has a formal derivative of the Wiener process.

However, when the results of the stochastic calculus were applied to other types phenomena, certain difficulties arose in the process of interpretation of stochastic differentials and approximation process. In many models, white noise process is explicitly introduced and the basic physical process in question is visualised as an approximation. Hence it is reasonable to expect some kind of a stability in the sense that the solutions that are obtained by approximating the white noise process should themselves approximate the process in question.

Ito stochastic calculus failed to satisfy this requirement of stability (see [McShane, 1974]). Moreover, in choosing the type of stochastic processes that we shall use us models of the noises we meet a dilema. On the one hand, there is no physical bases for considering an example considering any simple functions $W_j(t)$ exccept those of a rather simple structure. In fact, the noise input $W_j(t)-W_j(s)$ is measured be some sort of indicator and if this is mechanical it cannot move faster than the velocity of light, if it is electrical, it cannot suport more than some limited current or voltage difference without destruction and also some similiraties are in the financial modeling case.

In McShane's Calculus, the standard equations

$$(I) \qquad X^i(t,\omega) = X^i(0,\omega) + \int_0^t f^i(s,X(s,\omega))ds + \sum_{j=1}^r \int_0^t g_j^i(s,X(s,\omega))dW_j(s,\omega)$$

are replaced by what he calls a **canonical extension** (or canonical form or canonical system) of equation (I):

$$(II) \qquad X^i(t,\omega) = X^i(0,\omega) + \int_0^t f^i(s,X(s,\omega))ds +$$

$$\sum_{j=1}^r \int_0^t g_j^i(s,X(s,\omega))dW_j(s,\omega) + \frac{1}{2}\sum_{j,k=1}^r \int_0^t g_{j,k}^i(s,X(s,\omega))dW_j(s,\omega)dW_k(s,\omega)$$

in which

$$g_{j,k}^i(t,x,\omega) = \sum_{m=1}^n [\partial g_j^i(t,x,\omega)/\partial x^m] g_k^m(t,x,\omega)$$

$i=1,2,...,n;\ j,k=1,2,...,r;\ t\in[0,a];\ x\in\mathbf{R}^n$.

We are now able to describe the method by which we shall construct stochastic models of physical systems which in the physically realizable case of lipschitzian noises are known to satisfy the integral equation (I).

If $g_{j,k}^i(t,x,\omega)$ are functions defined for $t\in[0,a]$ and $x\in\mathbf{R}^n$ and bounded on bounded sets of (t,x), then the solution $X^i(t,\omega)$ of (I) is also a solution of (II) since the last integral vanishes for lipschitzian noises.

The McShane Calculus is better suited modeling dynamical phenomena described typically by McShane systems where $W_j(t,\omega)$ are noises processes.

McShane stochastic integral systems enjoy the following three important properties:

(i) The property of inclusiveness: the model must apply to systems in which the permitted noises are processes belonging to some family large enough to include processes with sample paths having lipschitzian property, all brownian motion processes, and such modifications as have proved convenient in applications;

(ii) The property of consistency: for lipschitzian noises, the solutions of the equations should coincide with the solutions of the equations that are normally believed to be applicable to physical systems;

(iii) The property of stability: the model must be such that if the noise process $W_j(t,\omega)$ is replaced by another permissible process $W_j^0(t,\omega)$ close to it, then the corresponding solutions $X^i(t,\omega)$, $X_0^i(t,\omega)$ are also close to each other (in the sense that an extreme degree of closeness corresponds to practical imposibility of distinguishing the process by means of available experimental procedures).

In section 2 we pursue the study of the solvability of a class of linear forward-backward stochastic differential equations of McShane type and we point out some drastic differences from the case of Itô type stochastic equations.

The approach developed in Section 2 is applied in Section 3 to a similar Black-Scholes type model in mathematical finance.

2 The main result

Consider the following forward-backward stochastic differential equations on $[0,T]$,

$$\begin{cases} dX(t) = [a(t)X(t) + b(t)]dt + [c(t)X(t) + d(t)]dW(t) \\ dY(t) = [f(t)X(t) + g(t)Y(t) + h(t)Z(t) + k(t)]dt + Z(t)dW(t) \\ X(0) = x_0, \ \ Y(T) = \alpha(X(T)) \end{cases} \tag{1}$$

where $T > 0$, $x_0 \in \mathbf{R}$ and $a, b, c, d, f, g, h, k : [0,T] \to \mathbf{R}$ are continuous functions, while $\alpha : \mathbf{R} \to \mathbf{R}$ is a function of class C^1. In (1), $(X(t), Y(t), Z(t))$ is a triplet of adapted stochastic processes on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathcal{P})$ such that $\{\mathcal{F}_t\}_{t\in[0,T]}$ is the natural filtration of a given stochastic process $\{W(t)\}_{t\in[0,T]}$, augmented with all $\mathcal{P}$-null sets. Throughout this paper, the process $\{W(t)\}_{t\in[0,T]}$ inducing the filtration is a finite variation process with continuous paths i.e. for almost all $\omega \in \Omega$ the sample path $t \to W(t,\omega)$ is continuous and of finite variation on $[0,T]$ as a particular noise of McShane type. For example, a process satisfying a.s. a Lipschitz condition

$$W(t,\omega) - W(s,\omega)| \leq L|t-s|, \ \ 0 \leq s \leq t \leq T$$

for some constant $L > 0$, is admissible. Let $C_{\mathcal{F}}[0,T]$ be the set of all $\{\mathcal{F}_t\}_{t\in[0,T]}$-progresively measurable continuous processes $X : [0,T]\times\Omega \to \mathbf{R}$ (that is, for almost all $\omega\in\Omega$ the sample paths $t \to X(t,\omega)$ is continuous on $[0,T]$), such that $E \sup\limits_{t\in[0,T]} |X(t)|^2 < \infty$. Observe that the space

$$M_{\mathcal{F}}[0,T] = C_{\mathcal{F}}[0,T] \times C_{\mathcal{F}}[0,T] \times C_{\mathcal{F}}[0,T]$$

is a Banach space under the norm

$$\|(X,Y,Z)\| = \{E \sup_{t\in[0,T]} |X(t)|^2 + E \sup_{t\in[0,T]} |Y(t)|^2 + E \sup_{t\in[0,T]} |Z(t)|^2\}^{\frac{1}{2}}.$$

Given $a,b,c,d,f,g,h,k \in C([0,T],\mathbf{R})$, $\alpha\in C^1(\mathbf{R},\mathbf{R})$, $x_0\in\mathbf{R}$, and the finite variation continuous process $\{W(t)\}_{t\in[0,T]}$ inducing the filtration on the probability space, a process $(X,Y,Z)\in M_{\mathcal{F}}[0,T]$ is called an *adapted solution* of (1) if the following holds for any $t\in[0,T]$, almost surely:

$$\begin{cases} X(t) = x_0 + \int\limits_0^t [a(s)X(s)+b(s)]ds + \int\limits_0^t [c(s)X(s)+d(s)]dW(s), \\ Y(t) = \alpha(X(T)) - \int\limits_t^T [f(s)X(s)+g(s)Y(s)+h(s)Z(s)+k(s)]ds - \\ \qquad - \int\limits_t^T Z(s)dW(s), \end{cases} \tag{2}$$

where the stochastic integrals are McShane type integrals (see [Protter, 1992] for an approach to this integral close in spirit to the original one by McShane [McShane, 1969]). In Section 3 we will give an example in mathematical finance that motivates the study of (1). Let us now prove the solvability of (1).

Theorem. The system (1) admits an adapted solution $(X,Y,Z)\in M_{\mathcal{F}}[0,T]$.

Proof. To show the existence of a solution we introduce a direct method, similar to the scheme developed in [Ma *et al.*, 1994] for Itô type forward-backward stochastic differential equations. We will prove that the following three-step scheme is realizable:

(A) let $\theta : [0,T]\times\mathbf{R}\to\mathbf{R}$ be the C^1-solution of the following first-order linear partial differential equation

$$\begin{cases} \theta_t + ([a(t)-c(t)h(t)]x + b(t) - d(t)h(t))\theta_x = \\ = g(t)\theta + f(t)x + k(t),\ t\in[0,T],\ x\in\mathbf{R}, \\ \theta(T,x) = \alpha(x),\ \ x\in\mathbf{R}; \end{cases} \tag{3}$$

(B) let $X \in C_{\mathcal{F}}[0,T]$ be the solution of the following forward stochastic differential equation of McShane type:

$$\begin{cases} dX(t) = [a(t)X(t) + b(t)]dt + \\ \qquad\qquad +[c(t)X(t) + d(t)]dW(t), \quad t \in [0,T], \\ X(0) = \; x_0; \end{cases} \tag{4}$$

(C) then X together with

$$Y(t) = \theta(t, X(t)), \quad Z(t) = \theta_x(t, X(t))(c(t)X(t) + d(t)), \quad t \in [0,T] \tag{5}$$

is an adapted solution to (1).

The special relations (5) among the components of the adapted solution $(X, Y, Z) \in M_{\mathcal{F}}[0,T]$ to (1) are suggested by the change of variables formula for McShane type stochastic integrals (see [McShane, 1974], p.146): if $X \in C_{\mathcal{F}}[0,T]$ solves (4), then

$$\begin{aligned} d\theta(t, X(t)) = [\theta_t(t, X(t)) + \theta_x(t, X(t))(a(t)X(t) + b(t))]dt + \\ +\theta_x(t, X(t))(c(t)X(t) + d(t))dW(t) \end{aligned}$$

and a comparison with the backward stochastic equation (for Y) in (2) confirms that the problem (3) for θ is precisely what is needed for the effectiveness of the solution scheme. Therefore, the existence part is proved if we show that steps (A) and (B) can be performed. Both problems can be explicitely solved. Indeed, the solution of (4) is (see [McShane, 1974], p.129-130)

$$X(t) = [x_0 + \Psi(t)]e^{\Phi(t)}, \quad t \in [0,T], \tag{6}$$

with

$$\Phi(t) = \int_0^t a(s)ds + \int_0^t c(s)dW(s), \quad t \in [0,T],$$

and

$$\Psi = \int_0^t e^{-\phi(s)}b(s)ds + \int_0^t e^{-\phi(s)}dsdW(s), \quad t \in [0,T].$$

On the other hand, the method of characteristics enables us [John, 1962] to write down the explicit C^1-solution of the Cauchy problem (3). However, taking into account the intricacy of the resulting formula, we refrain from further details- we shall do the full details of the solution in Section 3 for the choice of coefficients in (1) dictated by a model in mathematical finance. The proof of the theorem is completed.

The statement of the theorem leaves open the question of uniqueness. Our solution scheme was constructed in analogy with the four step scheme (see [Ma *et al.*, 1994]) for the Itô type problem (2)- case in which $\{W(t)\}_{t\in[0,T]}$ is

Brownian motion (a process with a.s. continuous sample paths but a.s. the sample paths are of unbounded variation functions [Protter, 1992]) and all stochastic integrals in (2) are of Itô type.

For the Itô type problem (2), uniqueness holds (see [Ma and Yong, 1999], p.82) so that it is not unreasonable to expect uniqueness in the McShane type problem (2) that we are investigating. However, let us note an essential difference in the two solution schemes (the four step scheme from [Ma and Yong, 1999] and our three step scheme) which indicates that the McShane type problem is not a perfect replicate to the Itô type problem. In both problems the forward stochastic differential equation are replaced by a forward stochastic differential equation coupled with a Cauchy problem for a partial differential equation: in the Itô type problem we have a parabolic partial differential equation while in the McShane scheme type problem we have a linear first order partial differential equation. For parabolic partial differential equations, time-reversibility is not to be expected whereas for linear first-order partial differential equations this is not an issue. Here lies an essential difference between the schemes adapted in the Itô type case, respectively in the McShane type case. An example illustrates that the uniqueness for the McShane type case isn't assured.

3 Applications

In this section we analyse a model in mathematical finance that motivates the study of forward-backward stochastic differential equations of McShane type.

Consider a market that contains one bond and one stock. Their prices at time t are denoted by $P(t)$ and $X(t)$, respectively. An investor trades continuously, the wealth of the investor at time t being denoted by $Y(t)$ and the amount of money invested into the stock at time t is denoted by $\pi(t)$, called portfolio, while the rest of the money at time t, $Y(t) - \pi(t)$, is put into the bond. In a stochastic model (model with uncertainly) one assumes that both prices are stochastic processes, defined on some filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P})$. The fact that both prices can only be determined by the information up to time t is expressed mathematically by requiring the processes $P(t),\ X(t)$ to be both adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$. We assume that the filtration is generated by a given continuous process $\{W(t)\}_{t\geq 0}$ with sample paths of bounded variation on compact intervals. If the market is assumed to be Markovian, that is, the interes rate $r(t)$ of the bond and the appreciation rate and volatility of the stock $b(t)$, respectively σ, are deterministic (the time-dependence is assumed to be continuous), then the prices are subject to the following system of stochastic differential equations

$$\begin{cases} dP(t) = r(t)P(t)dt, & \text{(bond)} \\ dX(t) = X(t)b(t)dt + \sigma X(t)dW(t), & \text{(stock)} \\ P(0) = \ 1, \ \ X(0) = x_0, \end{cases} \tag{7}$$

where $x_0 > 0$ is a constant. The change of wealth $dY(t)$ follows therefore the dynamics

$$dY(t) = \frac{\pi(t)}{X(t)} dX(t) + \frac{Y(t) - \pi(t)}{P(t)} dP(t). \tag{8}$$

An option with maturity date $T > 0$ is an $\mathcal{F}_T$-measurable random variable $\alpha(X(T))$, where $\alpha : \mathbf{R} \to \mathbf{R}$ is a function of class C^1. Let us rewrite (7)-(8) as

$$\begin{cases} P(t) = & e^{\int\limits_0^t r(s)ds}, \\ X(t) = & x_0 + \int\limits_0^t b(s)X(s)ds + \sigma \in_0^t t\, X(s)dW(s), \\ dY(t) = [\pi(t)b(t) + r(t)(Y(t) - \pi(t))]dt + \sigma\pi(t)dW(t), \end{cases}$$

for $t \in [0, T]$. The interaction between the investor's wealth/strategy and the stock price is described by the following forward-backward stochastic differential equations of McShane type

$$\begin{cases} X(t) = x_0 + \int\limits_0^t b(s)X(s)ds + \sigma \in_0^t t\, X(s)dW(s); t \in [0, T], \\ Y(t) = \alpha(X(T)) - \int\limits_t^T [r(s)Y(s) + (b(s) - r(s))\pi(s)]ds - \\ \qquad -\sigma \int\limits_t^T \pi(s)dW(s),\ t \in [0, T]. \end{cases} \tag{9}$$

The purpose of the investor is to find an adapted solution (X, Y, π) to (9); this amounts to choosing a strategy π allowing the realization of the option $Y(T) = \alpha(X(T))$.

The problem (9) is of type (2) so that we may apply our three step scheme developed in Section 2 to find an explicit solution. Relation (6) ensures that the solution of the equation for X in (9) is precisely

$$X(t) = x_0 e^{\sigma[W(t) - W(0)] + \int\limits_0^t b(s)ds}, \quad t \in [0, T]. \tag{10}$$

To find the explicit formula for the wealth $Y(t)$, we have to solve the problem (3), i.e.

$$\begin{cases} \theta_t + r(t)x\theta_x = r(t)\theta,\ t \in [0, T], \quad x \in \mathbf{R}, \\ \theta(T, x) = \alpha(x) \qquad\qquad x \in \mathbf{R}. \end{cases} \tag{11}$$

In accordance to the study pursued in Section 2, we apply the method of characteristics to solve (11). The characteristic curves are given by the system of ordinary differential equations (with parameter s)

$$\begin{cases} \dfrac{dt}{ds} = 1, \\ \dfrac{dx}{ds} = r(t)x, \\ \dfrac{d\theta}{ds} = r(t)\theta, \end{cases} \tag{12}$$

and the Cauchy data corresponds to (at $s = 0$)

$$t = T, \ \ x = \xi, \ \ \theta = \alpha(\xi). \tag{13}$$

The solution to (12)-(13) has the parametric representation

$$t = s + T, \quad x = \xi e^{\int\limits_0^s r(\tau+T)d\tau}, \quad \theta = \alpha(\xi) e^{\int\limits_0^s r(\tau+T)d\tau},$$

as it can be easily verified. Eliminating s, ξ we find for the C^1-solution of the Cauchy problem (11) the representation

$$\theta(t,x) = \alpha(xe^{\int\limits_t^T r(\tau)d\tau})e^{-\int limits_t^T r(\tau)d\tau} \tag{14}$$

since

$$s = t - T, \quad \xi = xe^{\int\limits_t^T r(\tau)d\tau}.$$

We can check directly that (14) solves (11).

As a consequence of our theorem, taking into account relations (5), (10) and (14), we find that a solution of the problem (9) is given by

$$\begin{cases} X(t) = x_0 e^{\sigma[W(t)-W(0)]+\int\limits_0^t b(s)ds}, & t \in [0,T], \\ Y(t) = \theta(t, X(t)), & t \in [0,T], \\ Z(t) = X(t)\theta_x(t, X(t)), & t \in [0,T]. \end{cases}$$

Remark. The model presented above is of Black-Scholes type because if we consider $b(t)$, $r(t)$ to be positive constants and the process $\{W(t)\}_{t\in[0,T]}$ to be a Brownian motion, interpreting (9) as an Itô type problem, we end up with a parabolic problem instead of (11): the Black- Scholes partial differential equation (see [Ma and Yong, 1999], p.227).

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