The Distributions of Stopping Times For Ordinary And Compound Poisson Processes With Non-Linear Boundaries: Applications to Sequential Estimation.

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Abstract. Distributions of the first-exit times from a region with non-linear upper boundary are discussed for ordinary and compound Poisson processes. Explicit formulae are developed for the case of ordinary Poisson processes. Recursive formulae are given for the compound Poisson case, where the jumps are positive, having continuous distributions with finite means. Applications to sequential point estimation are illustrated.
Keywords: Stopping times, sequential estimation, non-linear boundaries, compound Poisson processes.

1 Introduction

The distributions of stopping times for ordinary or compound Poisson processes when the boundaries are linear were studied in a series of papers by [Perry *et al.*, 1999a] [Perry *et al.*, 1999b] [Perry *et al.*, 2002a], [Perry *et al.*, 2002b], [Stadje and Zacks, 2003], [Zacks, 1991], [Zacks, 1997] and [Zacks *et al.*, 1999]. In particular, see the survey paper of [Zacks, 2005]. In the present paper we discuss the problem when the boundaries are non-linear. [Picard and Lefevre, 1996] studied crossing times of counting processes with non-linear lower boundaries, using pseudo-polynomials. We are developing a different approach for ordinary or compound Poisson processes with upper non-linear boundaries. In a recent paper by [Zacks and Mukhopadhyay, 2005], the theory presented here was applied to find the exact risk of sequential point estimators of the mean of an exponential distribution. Five different stopping rules with corresponding estimators were considered. By converting the problems to stopping times of an ordinary Poisson process, the boundaries were of two types: concave $B(t) = \gamma t^{\alpha}$, $0 < \alpha < 1$, and convex $B(t) = \gamma t^{\alpha}$, $\alpha > 1$. Explicit solutions were given there for the distributions of the estimators and their moments. When the distributions of the observed random variables are not exponential the situation is much more difficult. We assume that the observed random variables $X_1, X_2, \ldots$ are i.i.d. positive and that, for each $n \geq 1$, the sequence (n, S_n), where $S_n = \sum_{i=1}^{n} X_i$, is transitively sufficient. We apply the Poissonization method (see [Cesaroli, 1983], [Zacks, 1994]) to approximate the distribution of the stopping variable M in the sequential estimation by the distribution of a stopping time T. Here T is the first time that the compound Poisson Process $X(t) = S_{N(t)}$ crosses above an increasing boundary $B(t)$, where $\lim_{t\to\infty} B(t) = \infty$ and $\lim_{t\to\infty} B(t)/t = 0$.

While the derivation of the distributions of stopping times in the ordinary Poisson case is immediate, that for the compound Poisson process is complicated. We outline a solution by solving a sequence of related problems. In Section 2 we derive the distribution of a stopping time T, where an ordinary Poisson process $\{N(t), t \geq 0\}$ crosses an upper boundary $B(t)$. In Section 3 we discuss the problem when a compound Poisson process $X(t)$ crosses $B(t)$. In Section 4 we present an application to sequential estimation and some numerical results from [Zacks and Mukhopadhyay, 2005]. All lemmas and theorems are presented without formal proofs.

2 The Distribution of The First Crossing Time Of An Ordinary Poisson Process

Consider an ordinary Poisson process (OPP) $\{N(t), t \geq 0\}$ with $N(0) = 0$. This is a homogeneous process with intensity λ, $0 < \lambda < \infty$. For the properties of an OPP see [Kao, 1977].

Let $B(t)$ be strictly increasing, non-linear function of t, with $B(0) = 0$, $B(t) \nearrow \infty$ and $B(t)/t \to 0$ as $t \to \infty$. We are interested in the distribution of the stopping time

$$T = \inf\{t \geq t_k : N(t) \geq B(t)\}, \tag{1}$$

where for $l \geq k$ $t_l = B^{-1}(l)$. Since $B(t)$ is strictly increasing, $t_k < t_{k+1} < t_{k+2} < ....$ Notice that the distribution of $N(t_k)$ is Poisson(λt_k). Accordingly,

$$P(T = t_k) = 1 - P(k-1; \lambda t_k), \tag{2}$$

where $P(\cdot; \mu)$ is the cdf of Poisson(μ). We denote by $p(\cdot; \mu)$ the pdf of Poisson(μ).

Lemma 1 *For each λ, $0 < \lambda < \infty$,*

$$P_\lambda\{T < \infty\} = 1.$$

□

Define the defective probability function

$$g_\lambda(j; t) = P\{N(t) = j, T > t\}, \quad j = 0, 1, ... \tag{3}$$

for $t \geq t_k$. Since $T \geq t_k$ with probability one,

$$\begin{aligned} g_\lambda(j;t_k) &= p(j;\lambda t_k), \quad & j &= 0,...,k-1 \\ &= 0, & j &\geq k. \end{aligned} \tag{4}$$

Furthermore we have, for $t_{l-1} < t \leq t_l$, $l \geq k$, and $j = 0,...,l-1$

$$g_\lambda(j;t) = \sum_{i=0}^{j\wedge(l-2)} g_\lambda(i;t_{l-1})p(j-i;\lambda(t-t_{l-1})), \tag{5}$$

where $j \wedge (l-2) = \min(j, l-2)$. Thus, according to (5),

$$\begin{aligned} P_\lambda\{T > t\} &= \sum_{l=k+1}^{\infty} I(t_{l-1} < t \leq t_l) \sum_{j=0}^{l-1} g_\lambda(j;t) \\ &= \sum_{l=k+1}^{\infty} I(t_{l-1} < t \leq t_l) \sum_{j=0}^{l-2} g_\lambda(j;t_{l-1}) P(l-1-j;\lambda(t-t_{l-1})). \end{aligned} \tag{6}$$

Theorem 1 *The distribution function of T is absolutely continuous on (t_k, ∞) with density*

$$\Psi_T(t;\lambda) = \lambda \sum_{l=k+1}^{\infty} I(t_{l-1} < t < t_l) \cdot \sum_{j=0}^{l-2} g_\lambda(j;t_{l-1}) \cdot p(l-1-j;\lambda(t-t_{l-1})). \tag{7}$$

□

Theorem 2 *The r-th moment of T, $(r \geq 1)$, is*

$$\begin{aligned} E_\lambda\{T^r\} &= t_k^r(1 - P(k-1;\lambda t_k)) \\ &+ r! \sum_{l=k+1}^{\infty} t_{l-1}^r \sum_{j=0}^{l-2} g_\lambda(j;t_{l-1}) \sum_{i=0}^{r} \frac{1}{(r-i)!} \binom{l-1-j+i}{i} \cdot \\ &\cdot \frac{1}{(\lambda t_{l-1})^i}(1 - P(l-1-j+i;\lambda\Delta_l)), \end{aligned} \tag{8}$$

where $\Delta_l = t_l - t_{l-1}$. □

3 The Distribution of The First Crossing Time of a Compound Poisson Process

We consider here a compound Poisson process (CPP) with positive jumps. Accordingly, let $X_0 \equiv 0$, $X_1, X_2, ...$ be i.i.d. positive random variables having a common absolutely continuous distribution F, with density f. We assume that $F(0) = 0$.

Let $\{N(t), t \geq 0\}$ be an OPP, with intensity λ. We assume that $\{N(t), t \geq 0\}$ and $\{X_n, n \geq 1\}$ are independent. The CPP is $\{X(t), t \geq 0\}$, where

$$X(t) = \sum_{n=0}^{N(t)} X_n. \tag{9}$$

The distribution function of $X(t)$, at time t, is

$$H(x; t) = \sum_{n=0}^{\infty} p(n; \lambda t) F^{(n)}(x), \tag{10}$$

where $F^{(0)}(x) \equiv 1$ and $F^{(n)}(x)$ for $n \geq 1$ is the n-fold convolution

$$F^{(n)}(x) = \int_0^x f(y) F^{(n-1)}(x-y) dy. \tag{11}$$

The density of $H(x;t)$ for $x > 0$ is

$$h(x; t) = \sum_{n=1}^{\infty} p(n; \lambda t) f^{(n)}(x), \tag{12}$$

where $f^{(n)}$ is the n-fold convolution of f. For some $t > 0$ we are interested in the distribution of the stopping time

$$T_c = \inf\{t \geq t^* : X(t) \geq B(t)\}, \tag{13}$$

where $B(t)$ is the non-linear increasing boundary, as in Section 2. The distribution of T_c has an atom at $t = t_k$, given by

$$P\{T_c = t^*\} = 1 - H(B(t^*); t^*), \tag{14}$$

and

$$P\{T_c > t^*\} = H(B(t^*); t^*). \tag{15}$$

Moreover, see [Gut, 1988],

$$\lim_{t \to \infty} \frac{X(t)}{t} = \mu t,$$

where $\mu = E\{X_1\}$. Hence since $\dfrac{B(t)}{t} \to 0$ as $t \to \infty$ we obtain

Lemma 2 *For a CPP* $\{X(t), t \geq 0\}$

$$P\{T_c < \infty\} = 1. \tag{16}$$

□

Define the defective distribution

$$G(x;t) = P\{X(t) \le x, T_c > t\}. \tag{17}$$

Clearly,

$$P\{T_c > t\} = G(B(t);t). \tag{18}$$

Let $g(x;t)$ denote the defective density of $G(x;t)$. An explicit formula of $g(x;t)$ was derived by [Stadje and Zacks, 2003] for the case of a linear boundary $B(T) = \beta + t$. In the case of a non-linear boundary we follow the following steps.

First, define a sequence $\{B^{(m)}(t), m \ge 1\}$ of step-functions, such that $B^{(m)}(t) \le B^{(m+1)}(t)$ for all $m \ge 1$, all $0 \le t < \infty$, and such that $\lim_{m\to\infty} B^{(m)}(t) = B(t)$.

Second, define the stopping time

$$T_c^{(m)} = \inf\{t \ge t^* : X(t) \ge B^{(m)}(t)\}, \tag{19}$$

and correspondingly

$$G^{(m)}(x;t) = P\{X(t) \le x, T_c^{(m)} > t\}. \tag{20}$$

Notice that $\{T_c^{(m)} > t\} \subset \{T_c^{(m+1)} > t\}$, for all $m \ge 1$. Hence, by monotone convergence

$$\lim_{m\to\infty} G^{(m)}(x;t) = G(x;t). \tag{21}$$

Thus, we approximate $G(B(t);t)$ by $G^{(m)}(B^{(m)}(t);t)$ for m sufficiently large. For $m \ge 1$, let $\{t_l^{(m)}, l \ge 0\}$ be the end points of partition intervals of $[t^*, \infty)$, where

$$t_l^{(m)} = B^{-1}\left(B(t^*) + \frac{l}{m}\right), \quad l \ge 0. \tag{22}$$

The corresponding boundary $B^{(m)}(t)$ is given by the step-function

$$B^{(m)}(t) = \sum_{l=1}^{\infty} I\{t_{l-1}^{(m)} \le t < t_l^{(m)}\}\left(B(t^*) + \frac{l-1}{m}\right). \tag{23}$$

We develop now recursive formula for $G^{(m)}(x;t)$, $t \ge t^*$. Notice that $t_0^{(m)} = t^*$ for all $m \ge 1$.

Since $T_c^{(m)} \ge t^*$ for all $m \ge 1$,

$$G^{(m)}(x;t_0^{(m)}) = I\{x < B(t^*))H(x;t^*) + I(x \ge B(t^*))H(B(t^*),t^*). \tag{24}$$

Furthermore, since $B^{(m)}(t) \ge B(t^*)$ for all $t \ge t^*$ and $m \ge 1$,

$$G^{(m)}(x;t) = H(x;t), \quad x \le B(t^*), \tag{25}$$

for all $t > t^*$. In addition, for all $l \geq 1$,

$$G^{(m)}(x; t_l^{(m)}) = G^{(m)}(B^{(m)}(t_{l-1}^{(m)}); t_l^{(m)}), \quad \text{all} \quad x \geq B^{(m)}(t_{l-1}^{(m-1)}). \tag{26}$$

Finally, for every $l \geq 1$, $t_{l-1}^{(m)} < t \leq t_l^{(m)}$ and $x \leq B^{(m)}(t_{l-1}^{(m)})$,

$$G^{(m)}(x; t) = \int_0^x G^{(m)}(y; t_{l-1}^{(m)}) h(x - y; t - t_{l-1}^{(m)}) dy. \tag{27}$$

Thus, for each $m \geq 1$,

$$P\{T_c^{(m)} > t\} = \sum_{l=1}^{\infty} I\{t_{l-1}^{(m)} \leq t < t_l^{(m)}\} \cdot G^{(m)}(B^{(m)}(t_{l-1}^{(m)}); t). \tag{28}$$

Functionals of the distribution of $T_c^{(m)}$ can be derived from (28).

4 Application In Sequential Estimation: The Exponential Case.

Let $X_1, X_2, \ldots$ be i.i.d. random variables having an exponential distribution with mean β, $0 < \beta < \infty$. For estimating β consider the sequential stopping variable

$$M = \min\{m \geq k : m \geq (A/c)^{1/2} \bar{X}_m\}, \tag{29}$$

where A, c are positive constants and $\bar{X}_m = \frac{1}{m} \sum_{i=1}^{m} X_i$. For background information on this stopping rule and references see [Zacks and Mukhopadhyay, 2005]. At stopping we estimate β with the estimator $\bar{X}_M$. The corresponding risk is

$$R(\bar{X}_M, \beta) = A \cdot E\{(\bar{X}_M - \beta)^2\} + cE\{M\}. \tag{30}$$

[Zacks and Mukhopadhyay, 2005] applied the theory of Section 2 to evaluate exactly the functionals $E\{\bar{X}_M\}$ and $R(\bar{X}_M, \beta)$. This was done by considering the OPP $\{N(t), t \geq 0\}$ with intensity $\lambda = 1/\beta$. If we replace m and $m\bar{X}_m$, respectively, with $N(t)$ and t we obtain from (29) the related stopping time

$$T = \inf\{t \geq t_k : N(t) \geq \gamma t^{1/2}\}, \tag{31}$$

where $\gamma = (A/c)^{1/4}$ and $t_k = (k/\gamma)^2$. Here we have $M = N(T)$ and $\bar{X}_M = T/N(T)$. By slight modification of equation (8) we get the moments of $\bar{X}_M$. In Table 1 we present some exact values of $E\{M\}$, $E\{\bar{X}_M\}$ and $R(\bar{X}_M, \beta)$.

c	$\beta = 1$			$\beta = 1.25$		
	$E\{M\}$	$E\{\bar{X}_M\}$	$R(\bar{X}_M, \beta)$	$E\{M\}$	$E\{\bar{X}_M\}$	$R(\bar{X}_M, \beta)$
0.5	4.712	0.8663	4.1318	5.584	1.0739	5.4454
0.1	9.482	0.8757	2.2889	11.867	1.1145	2.9793
0.05	13.472	0.0915	1.7079	16.987	1.1500	2.1487
0.01	31.892	0.9597	0.7207	39.076	1.2124	0.8687
0.005	44.305	0.9742	0.4834	55.515	1.2249	0.5931

Table 1. Exact Values of $E\{M\}$, $E\{\bar{X}_M\}$ and $R(\bar{X}_m, \beta)$ for $A = 10$, $k = 3$.

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