Weighted Cramér-von Mises-type Statistics

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Abstract. We consider quadratic functionals of the multivariate uniform empirical process. Making use of Karhunen-Loève expansions of the corresponding limiting Gaussian processes, we obtain the asymptotic distributions of these statistics under the assumption of independent marginals. Our results have direct applications to tests of goodness of fit and tests of independence by Cramér-von Mises-type statistics.

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1 Introduction and Premiminaries.

1.1 Introduction.

In this paper, we survey some recent results ([14, 15, 13]) related to quadratic functionals of the form

$$\int_0^1 \cdots \int_0^1 t_1^{2\beta_1} \dots t_d^{2\beta_d} \alpha_{n,0}^2(t_1, \dots, t_d) dt_1 \dots dt_d, \tag{1}$$

where $\alpha_{n,0}$ is an appropriate version of the uniform empirical process on $[0,1]^d$ (see (36) in the sequel for explicit definitions). We first establish conditions on the $\beta_1, \dots, \beta_d$, under which the statistic in (1) converges to a quadratic functional of a Gaussian process, of the form

$$\int_0^1 \cdots \int_0^1 t_1^{2\beta_1} \dots t_d^{2\beta_d} \mathbf{B}_0^2(t_1, \dots, t_d) dt_1 \dots dt_d, \tag{2}$$

with $\mathbf{B}_0$ denoting a tied-down Brownian bridge. Second, we will characterize the distribution of the random variable in (2), through a Karhunen-Loève expansion of the corresponding weighted Gaussian process.

This problem has been initiated by Cramér [10] (see, e.g., Nikitin [26], Scott [32] and the references therein). In higher dimensions, we refer to Blum, Kiefer and Rosenblatt [6], Cotterill and Csörgő [8, 9], Deheuvels [13], Dugué [17, 18, 19], Hoeffding [20], Kiefer [24], Martynov [27], and Smirnov [33, 35, 34]. Quadratic functionals of Gaussian processes have been studied by Biane and Yor [5], Donati-Martin and Yor [16], Pitman and Yor [28, 29, 30, 31], and Yor [37, 38]. The results of Deheuvels and Martynov [14], and Deheuvels, Peccati and Yor [15], Deheuvels [13], give the core of the present survey paper. The theory of Bessel functions plays here an essential role and we refer to Bowman [7] and Watson [36] for details.

In §1.2 and 1.3, we give some preliminaries. We describe the univariate case in §2.1 and the multivariate case, with $d \geq 2$, in §2.2.

1.2 Some Preliminaries on Gaussian Processes.

Let $\{X(\mathbf{t}) : \mathbf{t} \in [0,1]^d\}$ denote a centered Gaussian process, with $d \geq 1$. We set $\mathbf{s} = (s_1, \ldots, s_d) \in \mathbb{R}^d$ and $\mathbf{t} = (t_1, \ldots, t_d) \in \mathbb{R}^d$, and set

$$R(\mathbf{s},\mathbf{t}) = \mathbb{E}\big(X(\mathbf{s})X(\mathbf{t})\big) \quad \text{for} \quad \mathbf{s},\mathbf{t} \in [0,1]^d. \tag{3}$$

We will are concerned with the *quadratic functional*

$$\int_{[0,1]^d} X^2(\mathbf{t})d\mathbf{t}, \tag{4}$$

where $d\mathbf{t}$ is the Lebesgue measure. We will work under the assumption that

$$0 < \mathbb{E}\Big(\int_{[0,1]^d} X^2(\mathbf{t})d\mathbf{t}\Big) = \int_{[0,1]^d} R(\mathbf{t},\mathbf{t})d\mathbf{t} < \infty. \tag{5}$$

The condition (5) entails that, almost surely, $X(\cdot) \in L^2\big([0,1]\big)$ belongs to the class of *Hilbert space valued centered Gaussian processes* (see, e.g., §10 in Lifshits [25]). By the Cauchy-Schwarz inequality, for each $\mathbf{s},\mathbf{t} \in [0,1]^d$,

$$R(\mathbf{s},\mathbf{t})^2 = \mathbb{E}\big(X(\mathbf{s})X(\mathbf{t})\big)^2 \leq \mathbb{E}\big(X(\mathbf{s})^2\big)\mathbb{E}\big(X(\mathbf{t})^2\big) = R(\mathbf{s},\mathbf{s})R(\mathbf{t},\mathbf{t}).$$

When combining this last inequality with (5), we obtain that

$$\|R\|_{L^2}^2 := \int_{[0,1]^d}\int_{[0,1]^d} R(\mathbf{s},\mathbf{t})^2 d\mathbf{s}d\mathbf{t} \leq \Big\{\int_{[0,1]^d} R(\mathbf{t},\mathbf{t})d\mathbf{t}\Big\}^2 < \infty, \tag{6}$$

so that $R \in L^2\big([0,1]^d \times [0,1]^d\big)$. Under (6), the Fredholm transformation $y(\cdot) \in L^2\big([0,1]^d\big) \to \widetilde{y}(\cdot)$, defined by

$$\widetilde{y}(\mathbf{t}) = \int_{[0,1]^d} R(\mathbf{s},\mathbf{t})y(\mathbf{s})d\mathbf{s} \quad \text{for} \quad \mathbf{t} \in [0,1]^d, \tag{7}$$

is a continuous linear mapping of $L^2\big([0,1]^d\big)$ onto itself. The condition (6) also implies the existence of a *convergent orthonormal sequence* [c.o.n.s.],

$\{\lambda_k, e_k(\cdot) : 1 \leq k < K\}$ with the following properties. $\{\lambda_k : 1 \leq k < K\}$ are positive constants and $K \in \{2, \ldots, \infty\}$ a possibly infinite index, with

$$\lambda_1 \geq \ldots \geq \lambda_k \geq \ldots > 0. \tag{8}$$

The $\{e_k(\cdot) : 1 \leq k < K\}$ are orthonormal in $L^2\big([0,1]\big)$, and fulfill

$$\int_{[0,1]^d} e_k(\mathbf{t}) e_\ell(\mathbf{t}) d\mathbf{t} = \begin{cases} 1 & \text{if} \quad k = \ell, \\ 0 & \text{if} \quad k \neq \ell. \end{cases}$$

The function R may be decomposed into the series

$$R(\mathbf{s}, \mathbf{t}) = \sum_{1 \leq k < K} \lambda_k e_k(\mathbf{s}) e_k(\mathbf{t}), \tag{9}$$

convergent in $L^2\big([0,1]^d\big)$. This entails that

$$\|R\|_{L^2} = \int_{[0,1]^d} \int_{[0,1]^d} R(\mathbf{s}, \mathbf{t})^2 d\mathbf{s} d\mathbf{t} = \sum_{1 \leq k < K} \lambda_k^2 < \infty. \tag{10}$$

The λ_k (resp. $e_k(\cdot)$) are the eigenvalues (resp. eigenfunctions) of the Fredholm operator (7), and fulfill the relations, for each $1 \leq k < K$,

$$\widetilde{e}_k(\mathbf{t}) = \int_{[0,1]^d} R(\mathbf{s}, \mathbf{t}) e_k(\mathbf{s}) d\mathbf{s} = \lambda_k e_k(\mathbf{t}). \tag{11}$$

The *Karhunen-Loève* [KL] decomposition of $X(\cdot)$, (see, e.g., Kac and Siegert [23, 22], Kac [21], Ash and Gardner [4], and Adler [2]) decomposes $X(\cdot)$ into

$$X(\mathbf{t}) = \sum_{1 \leq k < K} Y_k \sqrt{\lambda_k}\, e_k(\mathbf{t}), \tag{12}$$

where $\{Y_k : 1 \leq k < K\}$ are independent and identically distributed [i.i.d.] normal $N(0,1)$ random variables. Under (5), the series in (12) is convergent in mean square, since this condition is equivalent to

$$0 < \mathbb{E}\Big(\int_{[0,1]^d} X^2(\mathbf{t}) d\mathbf{t} \Big) = \sum_{1 \leq k < K} \lambda_k < \infty. \tag{13}$$

This, in turn, readily implies that, as $k \uparrow K$ with $k < K$,

$$\mathbb{E}\Big(\int_{[0,1]^d} \Big\{ X(\mathbf{t}) - \sum_{m=1}^{k} Y_m \sqrt{\lambda_m}\, e_m(\mathbf{t}) \Big\}^2 d\mathbf{t} \Big) = \sum_{m > k} \lambda_k \to 0.$$

The condition (5)–(13) is strictly stronger than (10). It implies that the quadratic functional (4) can be decomposed into the sum of the series

$$\int_{[0,1]^d} X^2(\mathbf{t}) d\mathbf{t} = \sum_{1 \leq k < K} \lambda_k Y_k^2. \tag{14}$$

The latter is almost surely convergent *if and only if* (5) holds. Therefore, we will assume, from now on, that this condition is satisfied.

1.3 A General Convergence Theorem.

With $R(\cdot,\cdot)$ as in (3), we consider independent replicæ $\xi_1(\cdot), \xi(2), \ldots$ of a general stochastic process $\xi(\cdot)$, fulfilling $(H.1\text{–}2\text{–}3)$ below.

$(H.1)$ $\quad \xi(\cdot) \in L^2\big([0,1]^d\big)$;

$(H.2)$ $\quad \mathbb{E}\big(\xi(\mathbf{t})\big) = 0$ for all $\mathbf{t} \in [0,1]^d$;

$(H.3)$ $\quad \mathbb{E}\big(\xi(\mathbf{s})\xi(\mathbf{t})\big) = R(\mathbf{s},\mathbf{t})$ for all $\mathbf{s},\mathbf{t} \in [0,1]^d$.

Under $(H.1\text{–}2\text{–}3)$ (see, e.g., Ex. 14, p. 205 in Araujo and Giné [3]), as $n \to \infty$, the convergence in distribution

$$\zeta_n(\cdot) := n^{-1/2} \sum_{i=1}^{n} \xi_i(\cdot) \ \xrightarrow{d}\ X(\cdot), \tag{15}$$

holds *if and only if* (5)–(13)) is satisfied, namely, when

$$\int_{[0,1]^d} \mathbb{E}\big(\xi^2(\mathbf{t})\big) d\mathbf{t} = \int_{[0,1]^d} R(\mathbf{t},\mathbf{t}) d\mathbf{t} < \infty.$$

We have therefore the following theorem.

Theorem 1 *Under* (5) *and* $(H.1\text{–}2\text{–}3)$*, we have, as* $n \to \infty$*, the convergence in distribution*

$$\int_{[0,1]^d} \zeta_n^2(\boldsymbol{t}) d\boldsymbol{t} \ \xrightarrow{d}\ \sum_{1 \le k < K} \lambda_k Y_k^2. \tag{16}$$

Proof. Under (5) (or equivalently (13)), it follows from (15) that

$$\int_{[0,1]^d} \zeta_n^2(\mathbf{t}) d\mathbf{t} \ \xrightarrow{d}\ \int_{[0,1]^d} X^2(\mathbf{t}) d\mathbf{t},$$

which, in turn, reduces (16) to a direct consequence of (15).□

Below, we provide some useful statistical applications of Theorem 1.

2 Weighted Empirical Processes.

2.1 The Univariate Case ($\boldsymbol{d = 1}$).

Let $U_1, U_2, \ldots$ be i.i.d. uniform $[0,1]$ random variables. For $n \ge 1$, set

$$F_n(t) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{\{U_n \le t\}}, \tag{17}$$

for the empirical distribution function [df] based upon $U_1, \ldots, U_n$, and let

$$\alpha_n(t) = n^{1/2}\big\{F_n(t) - t\big\} \quad \text{for} \quad t \in [0,1], \tag{18}$$

denote the *uniform empirical process*. Fix $\beta \in \mathbb{R}$, and set, for $n \geq 1$,

$$\xi_n(t) = t^\beta \big\{ 1\!\!1_{\{U_n \leq t\}} - t \big\} \quad \text{for} \quad t \in [0,1]. \tag{19}$$

We let $t^0 = 1$ for all $t \in \mathbb{R}$, when $\beta = 0$. In agreement with (15), (18), (19), and the notation of §1.3, we may write

$$\zeta_n(t) = n^{-1/2} \sum_{i=1}^{n} \xi_i(t) = t^\beta \alpha_n(t) \quad \text{for} \quad t \in [0,1]. \tag{20}$$

The assumptions (H.1–2–3) in §1.3 are fulfilled with R defined by

$$R(s,t) = s^\beta t^\beta \big\{ s \wedge t - st \big\} \quad \text{for} \quad s,t \in [0,1]. \tag{21}$$

For this choice of R, (5)–(13) hold if and only if

$$\int_0^1 t^{2\beta} \big\{ t(1-t) \big\} dt < \infty, \tag{22}$$

which is equivalent to $\beta > -1$. Now, since $s \wedge t - st$ is the covariance function of a standard Brownian bridge $\{B(t) : t \in [0,1]\}$, the kernel R in (21) is nothing else but the covariance function of the weighted Brownian bridge

$$X(t) = t^\beta B(t) \quad \text{for} \quad t \in (0,1]. \tag{23}$$

Deheuvels and Martynov [14] have given the KL decomposition of $X(\cdot)$ in (23) when $\beta \neq -1 \Leftrightarrow \nu = 1/(2(1+\beta)) > 0$. For $\nu \in \mathbb{R}$, we the first Bessel function (see, e.g.,§9.1.69 in Abaramowitz and Stegun [1]) is

$$J_\nu(x) = (\tfrac{1}{2}x)^\nu \sum_{k=0}^{\infty} \frac{(-\frac{1}{4}x^2)^k}{\Gamma(\nu+k+1)\Gamma(k+1)} \,. \tag{24}$$

Whenever $\nu > -1$, the positive zeros of J_ν are isolated and form an infinite increasing sequence $\{z_{\nu,k} : k \geq 1\}$, such that (see, e.g., Watson [36])

$$0 < z_{\nu,1} < z_{\nu,2} < \ldots, \tag{25}$$

and, as $k \to \infty$,

$$z_{\nu,k} = \big\{ k + \tfrac{1}{2}(\nu - \tfrac{1}{2}) \big\} + o(1). \tag{26}$$

Given this notation, Theorem 1.4 in [14] asserts that, whnever $\beta > -1$, the KL representation of $X(t) = t^\beta B(t)$ is given by

$$X(\mathbf{t}) = t^\beta B(t) = \sum_{k=1}^{\infty} Y_k \sqrt{\lambda_k}\; e_k(\mathbf{t}), \tag{27}$$

where $\{Y_k : k \geq 1\}$ are i.i.d. normal $N(0,1)$ random variables,

$$\lambda_k = \Big\{ \frac{2\nu}{z_{\nu,k}} \Big\}^2, \quad k = 1, 2, \ldots, \tag{28}$$

and

$$e_k(t) = t^{\frac{1}{2\nu} - \frac{1}{2}} \Big\{ \frac{J_\nu(z_{\nu,k} t^{\frac{1}{2\nu}})}{\sqrt{\nu}\, J_{\nu-1}(z_{\nu,k})} \Big\} \quad \text{for} \quad 0 < t \leq 1. \tag{29}$$

Refer to Deheuvels and Martynov [14] for details. We get the theorem:

Theorem 2 *For any $\beta > -1$, setting $\nu = 1/(2(1+\beta))$, we have, as $n \to \infty$, the convergence in distribution*

$$\int_0^1 t^{2\beta} \alpha_n^2(t) dt \ \xrightarrow{d}\ \int_0^1 t^{2\beta} B^2(t) dt = \sum_{k=1}^\infty \Big\{ \frac{2\nu}{z_{\nu,k}} \Big\}^2 Y_k^2, \tag{30}$$

where $\{Y_k : k \geq 1\}$ is an i.i.d. sequence of normal $N(0,1)$ random variables.

Proof. In view of (28)–(29), it is a direct consequence of Theorem 1.□

2.2 The Multivariate Case ($d \geq 2$).

We now let $d \geq 2$. When $\mathbf{s} = (s_1, \ldots, s_d) \in \mathbb{R}^d$ and $\mathbf{t} = (t_1, \ldots, t_d) \in \mathbb{R}^d$, we denote by $\mathbf{s} \leq \mathbf{t}$ the fact that $s_j \leq t_j$ for $j = 1, \ldots, k$, and set, accordingly,

$$\mathbf{s} \wedge \mathbf{t} = \big(s_1 \wedge t_1, \ldots, s_d \wedge t_d \big).$$

Letting $\mathbf{U} = (U(1), \ldots, U(d)) \in [0,1]^d$ be uniformly distributed on $[0,1]^d$, we let $\mathbf{U}_n = (U_n(1), \ldots, U_n(d)) \in [0,1]^d$, $n = 1, 2, \ldots$ be i.i.d. replicæ of $\mathbf{U}$. For each $n \geq 1$, the empirical df based upon $\mathbf{U}_1, \ldots, \mathbf{U}_n$ is denoted by

$$F_n(\mathbf{t}) = \frac{1}{n} \sum_{i=1}^n \mathbb{I}_{\{\mathbf{U}_i \leq \mathbf{t}\}}, \tag{31}$$

We denote by

$$F(\mathbf{t}) = \mathbb{P}\big(\mathbf{U} \leq \mathbf{t}\big) = \prod_{j=1}^d t_j, \tag{32}$$

the (*exact*) distribution function of $\mathbf{U}$, and set

$$\alpha_n(\mathbf{t}) = n^{1/2}\big(F_n(\mathbf{t}) - F(\mathbf{t})\big) \quad \text{for} \quad \mathbf{t} \in [0,1]^d, \tag{33}$$

for the corresponding uniform empirical process. Making use of §1.3, we obtain that the following convergence in distribution holds. As $n \to \infty$,

$$\alpha_n(\cdot) \ \xrightarrow{d}\ \mathbf{B}(\cdot), \tag{34}$$

where $\{\mathbf{B}(\mathbf{t}) : \mathbf{t} \in [0,1]^d\}$ is a standard multivariate Brownian bridge. Namely, $\mathbf{B}(\cdot)$ is a centered Gaussian process, with covariance function

$$\begin{aligned}\mathbb{E}\big(\mathbf{B}(\mathbf{s})\mathbf{B}(\mathbf{t})\big) &= \mathbb{E}\big(\alpha_n(\mathbf{s})\alpha_n(\mathbf{t})\big) \\ &= \prod_{j=1}^{d} \big\{s_j \wedge t_j\big\} - \prod_{j=1}^{d} \big\{s_j t_j\big\}. \end{aligned} \tag{35}$$

The KL decomposition of $\mathbf{B}(\cdot)$, with covariance function as in (35), is not known explicitly for $d \geq 2$. A more tractable *tied-down* empirical process $\alpha_{n,0}(\cdot)$ is as follows. Set

$$\begin{aligned}\alpha_{n,0}(\mathbf{t}) &= \alpha_n(\mathbf{t}) - \sum_{1 \leq j \leq d} t_j \ \alpha_n(t_1, \ldots, t_{j-1}, 1, t_{j+1}, \ldots, t_d) \\ &+ \sum_{1 \leq j < \ell \leq d}^{d} t_j t_\ell \ \alpha_n(t_1, \ldots, t_{j-1}, 1, t_{j+1}, \ldots, t_{\ell-1}, 1, t_{\ell+1}, \ldots, t_d) \\ &+ \ldots + (1)^d t_1 \ldots t_d \ \alpha_n(1, \ldots, 1). \end{aligned} \tag{36}$$

In (36), $\alpha_n(1, \ldots, 1) = 0$, but this term is stated for convenience. In view of §1.3, we obtain the following convergence in distribution. As $n \to \infty$,

$$\alpha_{n,0}(\cdot) \ \stackrel{d}{\to} \ \mathbf{B}_0(\cdot), \tag{37}$$

where $\{\mathbf{B}_0(\mathbf{t}) : \mathbf{t} \in [0,1]^d\}$ is a tied-down multivariate Brownian bridge. Namely, $\mathbf{B}_0(\cdot)$ is a centered Gaussian process, with covariance function

$$\mathbb{E}\big(\mathbf{B}_0(\mathbf{s})\mathbf{B}_0(\mathbf{t})\big) = \prod_{j=1}^{d} \big\{s_j \wedge t_j - s_j t_j\big\}. \tag{38}$$

We have the following easy consequence of the results of Deheuvels and Martynov [14] (see also Deheuvels, Peccati and Yor [15]).

Theorem 3 *Let $\beta_1, \ldots, \beta_d$ be constants such that $\beta_j > -1$ for $j = 1, \ldots, d$. Set $\nu_j = 1/(2(1+\beta_j)) > 0$ for $j = 1, \ldots, d$. Then, the Karhunen-Loève decomposition of the centered Gaussian process*

$$X(\boldsymbol{t}) = t_1^{\beta_1} \ldots t_d^{\beta_d} \mathbf{B}_0(\boldsymbol{t}) \quad for \quad \boldsymbol{t} \in (0,1]^d, \tag{39}$$

is given by

$$X(\boldsymbol{t}) = \sum_{k_1=1}^{\infty} \ldots \sum_{k_d=1}^{\infty} \sqrt{\lambda_{k_1,\ldots,k_d}} \, Y_{k_1,\ldots,k_d} \ e_{k_1,\ldots,k_d}(\boldsymbol{t}), \tag{40}$$

where

$$\lambda_{k_1,\ldots,k_d} = \prod_{j=1}^{d}\Big\{\frac{2\nu_j}{z_{\nu_j,k_j}}\Big\}^2 =: \prod_{j=1}^{d}\mathcal{L}(\nu_j,k_j), \tag{41}$$

and

$$\begin{aligned} e_{k_1,\ldots,k_d}(\boldsymbol{t}) &= \prod_{j=1}^{d}\left[t_j^{\frac{1}{2\nu_j}-\frac{1}{2}}\left\{\frac{J_{\nu_j}(z_{\nu_j,k}\,t_j^{\frac{1}{2\nu_j}})}{\sqrt{\nu_j}\,J_{\nu_j-1}(z_{\nu_j,k})}\right\}\right] \\ &=: \prod_{j=1}^{d}\mathcal{E}(\nu_j,t_j). \end{aligned} \tag{42}$$

Proof. By (38) the covariance function of $X(\mathbf{t})$ in (39) is given by

$$R(\mathbf{s},\mathbf{t}) = \prod_{j=1}^{d} s_j^{\beta_j} t_j^{\beta_j}\{s_j\wedge t_j - s_j t_j\} =: \prod_{j=1}^{d}\mathcal{R}(s_j,t_j). \tag{43}$$

Therefore, via (28)–(29), $\lambda_{k_1,\ldots,k_d}$ is an eigenvalue of the Fredholm operator (7) pertaining to $e_{k_1,\ldots,k_d}(\cdot)$. To conclude, we show that *all* eigenvalues are so obtained. For this, we combine (10) with (43), to write that

$$\begin{aligned} &\int_{[0,1]^d}\int_{[0,1]^d} R(\mathbf{s},\mathbf{t})^2 d\mathbf{s}d\mathbf{t} = \prod_{j_1=1}^{\infty}\cdots\prod_{j_d=1}^{\infty}\int_0^1\int_0^1 \mathcal{R}(s_j,t_j)^2 ds_j dt_j \\ &= \prod_{j_1=1}^{\infty}\cdots\prod_{j_d=1}^{\infty}\Big\{\sum_{k_j=1}^{\infty}\mathcal{L}(\nu_j,k_j)^2\Big\} = \sum_{k_1=1}^{\infty}\cdots\sum_{k_d=1}^{\infty}\lambda^2_{k_1,\ldots,k_d}. \end{aligned}$$

This shows that there is no other remaining eigenvalue of (7).□

The next theorem is an easy consequence of the preceding results.

Theorem 4 *Let $\beta_1,\ldots,\beta_d$ be constants such that $\beta_j > -1$ for $j=1,\ldots,d$. Set $\nu_j = 1/(2(1+\beta_j)) > 0$ for $j=1,\ldots,d$. Then, we have, as $n\to\infty$,*

$$\begin{aligned} &\int_{[0,1]^d} t_1^{2\beta_1}\ldots t_d^{2\beta_d}\alpha_{n,0}^2(\boldsymbol{t})d\boldsymbol{t} \overset{d}{\to} \int_{[0,1]^d} t_1^{2\beta_1}\ldots t_d^{2\beta_d}\mathbf{B}_0^2(\boldsymbol{t})d\boldsymbol{t} \\ &= \sum_{k_1=1}^{\infty}\cdots\sum_{k_d=1}^{\infty}\Big\{\prod_{j=1}^{d}\Big\{\frac{2\nu_j}{z_{\nu_j,k_j}}\Big\}^2\Big\}Y^2_{k_1,\ldots,k_d}, \end{aligned} \tag{44}$$

where $\{Y_{k_1,\ldots,k_d} : k_1\ge 1,\ldots,k_d\ge 1\}$ is an i.i.d. array of normal $N(0,1)$ random variables.

The limiting distribution in Theorem 4 coincides with that of the Blum-Kiefer-Rosenblatt statistic (see, e.g., [6]), when $d=2$ and $\beta_1=\ldots=\beta_d=0$.

Conclusion. For $d \geq 2$, the eigenvalues $\lambda_{k_1,\ldots,k_d}$ in the KL decomposition (41)–(42) are multiple. This renders the numerical computation of the limit distribution of the test statistic in (44) more delicate than in the univariate case. This problem will be investigated elsewhere.

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