Nonparametric laws in the performance evaluation of the reliable, unreliable and renewal systems

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Abstract. In this paper we consider the performance evaluation problem in a queuing system GI/GI/1, a system with two units of reparable elements and a queuing system with an unreliable server and service repetition, using nonparametric distributions (consider IFR, NBU, DFR and NWU classes). We considered the qualitative properties of the inter-arrival, the repair and the service time, respectively. And presente bounds for the mean stationary waiting time, the mean time of life system and the blocking time in the system, respectively. These bounds are programmed and the characteristics are simulated in order to supplement the work carried out in [Adjabi *et al.*, 2004] and [Lagha and Adjabi, 2004].

Keywords: Nonparametric distribution, performance evaluation, queuing system, bounds, simulation.

1 Introduction

In this paper we are interested to the use of the qualitative properties of distributions for the characterstics evaluation in three systems, it acts a queuing system GI/GI/1, a renewal system and a queuing system with an unreliable server. The distributions considered are those of the inter-arrival, the reparation and the service time, respectively.
The aim of this paper is to calculate the characteristics by simulation in order to verify there membership to the interval delimited by the bounds established in [Adjabi *et al.*, 2004] and [Lagha and Adjabi, 2004]. These bounds are presented in the section 2, 3 and 4, respectively to the considered systems. Whereas the characteristics are simulated in the section 5. The results are interpreated in the section 6.

2 Queuing system GI/GI/1

Consider two systems $A_1/B_1/1$ (known as original) and $A_2/B_2/1$ (known as approximation system). We consider the following notations:

m_{W_i} : means of waiting time in $A_i/B_i/1$ system, $i = 1, 2$.
$A_i, i = 1, 2$: inter-arrival time distribution.
$B_i, i = 1, 2$: service time distribution.
m_B, m_A : means of service and inter-arrival times.
EB^2 : second moment of service time.
σ_B^2, σ_A^2 : variances of service and inter-arrival times.
$C_a = \sigma_A/m_A$: coefficient of variation of the inter-arrival times.
$\rho_i = m_B/m_{A_i}$: intensity of the traffic in the system $i, i = 1, 2$.

Under the external monotonicity property (theorem 5.2.1 in [Stoyan, 1983]),

$$A_2 \leq_{cv} A_1 \quad \text{and} \quad B_1 \leq_c B_2 \ , \tag{1}$$

we obtain the comparison $m_{W_1} \leq m_{W_2}$. Where $\leq_c$ ($\leq_{cv}$) indicates the convex (concave) ordering.
Suppose that $B_1 = B_2 = B$ a general service distribution and A_1 being a nonparametric inter-arrival distribution (IFR or NBU), its lower bound is given from the following table [Sengupta, 1994]:

Class	upper Bound	lower Bound
IFR	$\overline{F}(x) \leq \begin{cases} 1 & \text{if } x < m_r^{1/r} \\ \delta_x & \text{if } x > m_r^{1/r} \end{cases}$ where $\int_0^1 r y^{r-1} \delta_x^y dy = \frac{m_r}{x^r}$	$\overline{F}(x) \geq \begin{cases} \inf\limits_{0 \leq \beta \leq x} e^{-\alpha} & \text{if } x < m_r^{1/r} \\ 0 & \text{if } x > m_r^{1/r} \end{cases}$ where $\int_0^\infty (\beta + \frac{x-\beta}{\alpha} z)^r e^{-\alpha} dz = m_r$
NBU	$\overline{F}(x) \geq \begin{cases} 1 & \text{if } x < m_r^{1/r} \\ \delta_x & \text{if } x \geq m_r^{1/r} \end{cases}$ where $\int_0^1 r y^{r-1} \delta_x^y dy = \frac{m_r}{x^r}$	$\overline{F}(x) \geq \begin{cases} \delta_x & \text{if } x < m_r^{1/r} \\ 0 & \text{if } x \geq m_r^{1/r} \end{cases}$ where $\sum_{j=0}^\infty \delta_x^j [(j+1)^r - j^r] = \frac{m_r}{x^r}$
DFR	$\overline{F}(x) \leq \begin{cases} e^{\frac{-rx}{x_0}} & \text{if } x < m_r^{1/r} \\ (\frac{x_0}{x})^r e^{-r} & \text{if } x \geq m_r^{1/r} \end{cases}$ where $x_0 = r[\frac{m_r}{\Gamma(r+1)}]^{1/r}$	$\overline{F}(x) \geq 0$
NWU	$\overline{F}(x) \leq \delta_x$ where $\sum_{j=1}^\infty \delta_j^x [j^r - (j-1)^r] = \frac{m_r}{x^r}$	$\overline{F}(x) \geq 0$

Table 1: Bounds on $\overline{F}(x)$ (based on r moment m_r) in various cases.

Using this property (for A_1 being IFR or NBU distribution, see Table 1.) and the relation (1), we presente the upper bounds for the mean stationary waiting time in two cases.
• **In the IFR case**, the upper bound associated is given by the following

relation :

$$\frac{\sigma_{A_1}^2 + \sigma_B^2}{2m_{A_1}(1-\rho_1)} - 1/2m_{A_1}(\rho_1 + C_{a1}^2) \leq m_{W_1} \leq \frac{EB^2}{\frac{2}{\alpha}(1 - e^{-1} - \rho_1)}$$

here $\alpha = \left[\frac{\Gamma(r+1)}{m_r}\right]^{1/r}$.

• **In the NBU case**, the upper bound is given by the following relation :

$$\frac{\sigma_{A_1}^2 + \sigma_B^2}{2m_{A_1}(1-\rho_1)} - 1/2m_{A_1}(\rho_1 + 1) \leq m_{W1} \leq \frac{EB^2}{m_{A_1}[1 - e^{-1} - 2\rho_1]}$$

Remark 1
• *The lower bounds in the IFR and NBU cases are proposed by Stoyan [Stoyan, 1983].*
• *See [Adjabi* et al.*, 2004] for demonstrations.*

3 Renewal theory

Consider a system with two units of reparable elements ξ_1 and ξ_2. At the moment $t = 0$ the element ξ_1 function until there failure at the date $t = X_0$ where repair starts with to be carried out on this element and takes a time equalize with Y_1 whereas ξ_2 starts to work until its failure with the date $t = X_1$. If $X_1 \leq Y_1$,the system stops with date $X_0 + X_1$. If not ξ_1 still function at the date X_0+X_1 whereas the repair of ξ_2 is started. The operating time $X_0, X_1, \ldots$, are supposed iid and independent of times of successive repairs Y_1, Y_2, $\ldots$, which are too iid and with the mean m_r of order, $r > 1$. We defined $N = inf\{n : X_n < Y_n\}$ life time for the system is there r.v. $T = X_0 + X_1 + \cdots + X_N$. So now them X_i is exponentially distributed of parameter λ and arbitrary repair time function R (cumulative distribution function of Y) is a nonparametric distribution (IFR, NBU, DFR or NWU).

Proposition 1 *Consider two systems as described above having for function of repair time distribution R_i, $i = 1, 2$ and λ_i, $i = 1, 2$, indicating parameters of the operating time, respectively. Given the following condition [Stoyan, 1983] :*

$$\lambda_1 \leq \lambda_2 \quad \textit{and} \quad R_1 <_L R_2, \tag{2}$$

the comparison between the mean time of life systems is as $ET_1 \geq ET_2$, where $<_L$ indicates the Laplacien order.

Using the lower bound of R_1 (see the Table 1.) and the relation (2), the upper bounds of the mean time of life system are given by the following relation (see [Adjabi *et al.*, 2004] for demonstrations):

• **IFR case**

$$1 + (1 - e^{-\beta})^{-1} \leq ET_1 \leq \frac{1}{\lambda}\Big[1 + \frac{1 + \beta^{-1}}{1 - e^{-(1+\beta)(\Gamma(r+1))^{1/r}}}\Big]$$

with $\alpha = \left[\frac{\Gamma(r+1)}{m_r}\right]^{1/r}$, $\beta = \frac{\lambda}{\alpha}$ and $r > 0$.

• **NBU case**

$$1 + (1 - e^{-\beta})^{-1} \leq \lambda ET_1 \leq 1 + \frac{\beta}{\beta + e^{-\beta} - 1}, \quad \beta = \lambda m_1$$

Using the upper bound for R_1 (see the Table 1.)and the relation (2), the lower bounds of the mean time of life system are given by the following relation (see [Adjabi *et al.*, 2004] for demonstrations):

• **DFR case**

$$1 + \frac{\theta + r}{\theta(1 - e^{-(r+\theta)}) + (r+\theta)e^{-r}\theta^r \int_\theta^\infty x^{-r}e^{-x}dx} \leq \lambda ET_1 \leq 1 + (1 - e^{-\lambda m_1})^{-1}$$

where $\theta = \lambda x_0$, $x_0 = r\left[\frac{m_r}{\Gamma(r+1)}\right]^{1/r}$ and $r > 0$.

• **NWU case**

$$1 + \frac{1}{1 - \theta e^\theta \int_\theta^\infty x^{-2}e^{-x}dx} \leq \lambda ET_1 \leq 1 + (1 - e^{-\lambda m_1})^{-1}, \quad \theta = \lambda m_1$$

Remark 2

• *The lower bound presented in IFR and NBU cases is proposed by Stoyan [Stoyan, 1983].*

• *This bound became the upper one in the NWU and DFR cases.*

4 Unreliable queuing system

Consider a single-server queuing system with an unreliable server and service repetition. The total time taken by a customer from the instant he enters for service to the instant when he ends his service is called the blocking time which can be represented by:

$$Z_\lambda = X.1_{\{X \leq Y\}} + (Y + Z_\lambda^*).1_{\{X > Y\}}, \quad \lambda > 0. \tag{3}$$

Where X, Y and Z_λ are independent non-negative random variables, with cumulative distribution functions (cdf) denoted by $G(t), R(t)$ and $F(t)$, respectively. X is the service time with free interruption, Y is the server failure time and is assumed to have exponential distribution with mean $1/\lambda$. So $\overline{R}(t) = 1 - R(t) = e^{-\lambda t}$, $0 \leq t \leq \infty$.

Z_λ^* has the same distribution as Z_λ (denoted by $Z_\lambda^* \overset{d}{=} Z_\lambda$), and $1_{\{X \leq Y\}}$ is the indicator function of the event $\{X \leq Y\}$.

Consider the cdf G of X being a nonparametric repair distribution (IFR, NBU, DFR or NWU), its lower or upper bounds are given from the Table 1. To gather with the following Lemma, the bounds of the mean blocking time in the system EZ_λ are established (see [Lagha and Adjabi, 2004] for demonstrations). Let EX^r denote the r th moment of the r.v. X.

Lemma 1 *Suppose that X is not degenerate at point zero and Z defined as (3), so*

$$EZ = E(min(X,Y))/p \ ,$$

where $p = P(X \leq Y) = \int_0^{\infty} G(t)dR(t)$ *so* $E(min(X,Y)) = \int \overline{G}(t)e^{-\lambda t}\, dt$.

Using the corresponding lower bounds for G (see the Table 1.) and the above Lemma the lower bounds of the mean blocking time in the system are given in the following cases :

• **IFR case**

$$EZ_\lambda \geqslant x_0\Big[\frac{1-e^{-1-\lambda x_0}}{1+x_0\lambda e^{-1-\lambda x_0}}\Big],\ x_0 = EX$$

• **NBU case**

$$EZ_\lambda \geqslant \frac{\beta + e^{-\beta} - 1}{\lambda(1-e^{-\beta})},\ \beta = \lambda x_0$$

The upper one are given in the remainder cases considered :

• **DFR case**

$$EZ_\lambda \leq \frac{x_0(e^r - e^{-\lambda x_0}) + (r+\lambda x_0)x_0^r I_r}{re^r + \lambda x_0 e^{-\lambda X_0} - \lambda(r+\lambda x_0)x_0^{-r} I_r}$$

where $I_r = \int_{x_0}^{+\infty} t^{-r}e^{-\lambda t}\, dt$, $x_0 = r\Big[\frac{EX^r}{\Gamma(r+1)}\Big]^{1/r}$ and $r > 0$.

• **NWU case**

$$EZ_\lambda \leq \frac{x_0 e^{\beta} I_1}{1-\beta e^{\beta} I_1},\quad \beta = \lambda x_0 \ \text{ and } \ I_1 = \int_{x_0}^{+\infty} t^{-1}e^{-\lambda t}dt$$

Remark 3

• *The complex integral I_r is convergent and simulated (in the following section) to calculate the bounds.*

5 Bounds Computation

Consider in this section some parametric distributions to calculate the bounds given above (for three systems) and simulate characteristics. The application is worked in MATLAB environment.

The results are given in the following tables for three considered problems, respectively.

• **Queuing System GI/GI/1**

System	lower Bound	upper Bound	Simulation
$E_{(4,2)}/E_{(1,3)}/1$	0	0.11936	0.0097321
$E_{(4,3)}/E_{(2,5)}/1$	0	0.27099	0.021166

$E_{(2,3.5)}/W_{(1,4)}/1$	0.083333	0.56199	0.11258
$W_{(3,1.5)}/W_{(1,4)}/1$	0	0.4948	0.051066
$E_{(3,1)}/M_{\mu=2}/1$	0	0.17904	0.018002
$M_{\lambda=0.5}/E_{(1,2)}/1$	0.16667	0.32712	0.18617
$M_{\lambda=0.7}/W_{(1,4)}/1$	0.05003	0.095708	0.050896
$M_{\lambda=1.5}/M_{\mu=2}/1$	0.3	0.37359	0.3
$IFR/M_{\mu=1.2}/1$	0.12987	2.5541	0.32971
$IFR/IFR/1$	0	0.3069	0

Table 2: Bounds and simulation of the waiting average time

• **Renewal system**

Consider for application, the rth moment of the r.v. Y ($r = 1, 5$ and 10).

$exp(\lambda)/R(t)$ model	r order	lower bound	upper bound	simulation
$\lambda = 2/E_{(2,3)}$	1	1.179	1.469	1.2839
	5		1.5305	
	10		3.1114	
$\lambda = 1.2/Exp(1.1)$	1	2.0882	2.7609	2.4071
	5		2.5091	
	10		2.5002	
$\lambda = 1.5/W_{(2,4)}$	1	1.9302	4.2973	3.1272
	5		5.183	
	10		6.355	
$\lambda = 3/W_{(0.5,3)}$	1	0.83127	1.1805	1.0549
	5	0.70901		
	10	0.68856		
$\lambda = 2/IFR$	1	1.0034	1.1015	1.0088
	5		1.2045	
	10		1.3233	
$\lambda = 1.2/W_{(0.8,1.5)}$	1	2.5339	2.5962	2.5132
	5	2.3289		
	10	2.2295		

Table 3: Bounds and simulation of life average time

- **Unreliable system**

Failure rate	$\lambda = 2$	$\lambda = 3$	$\lambda = 3.5$	$\lambda = 1.6$
Service time	$E_{(2,4)}$	$Exp(4)$	$W_{(2,3)}$	IFR
Lower bound	0.3808	0.18274	0.63459	0.58863
Upper bound	x	x	x	x
Simulation	1.5040	0.8947	0.6611	2.4588

Table 4: Bounds and simulation of the blocking average time

6 Results interpretation

In the first system, we remark that the characteristic value obtained by the simulation belongs to the interval delimited by the lower and upper bounds presented in the section 2 and prooved in [Adjabi *et al.*, 2004]. This let us think that these bounds are accepted. Moreover the characteristic value seems to be much closer to the lower bound than the upper one.

Remark in the second system that the characteristic value obtained by simulation belongs to the proposed interval délimited by the bounds presented in the section 3. In the models where the repair time distribution is IFR, the upper bound is an increased function of r but the lower one does not depend on r. In the models where the repair time distribution is DFR, the lower bound is an increased function of r but the upper one does not depend on r.
We remark in addition that, the computed value by simulation turns around 1 when $\lambda \geq 2$ whereas it largely exceeds 1 when λ turns around 1.

In the last system, we considered the IFR and UBU cases for application. So we have only the lower bound to calculate and the values obtained by simulation are finite and higher then those of lower bounds. The "x" means no upper bound is calculated.

7 Conclusion

In this work we considered the performance evaluation problem in the queuing system GI/GI/1 (section 2), a renewal system (section 3) and an unreliable system (section 4), using nonparametric properties of distributions (IFR, NBU, DFR or NWU class). By comparison between distributions with stochastic orders ($<_c, <_{cv}$ and $<_L$), bounds are presented for considered systems. The characteristics bounds obtained are for: mean waiting time, the mean life time and the mean blocking time, respectively.
These systems are simulated in order to supplement the works of [Adjabi *et*

al., 2004] and [Lagha and Adjabi, 2004] to verify the acceptance of the proposed bounds (section 5). This verification is established by the application worked in MATLAB environment.
The bounds presented in this paper can be used for other distributions.
Example: using the following relations

$$IFR \rightarrow IFRA \rightarrow NBU \text{ and } DFR \rightarrow DFRA \rightarrow NWU$$

for IFRA, DFRA,...

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