Input Control in Fuzzy Non-Homogeneous Markov Systems

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Abstract. Certain aspects of input control of a non-homogeneous Markov System (NHMS) using fuzzy set theory and fuzzy reasoning are presented in this paper. This is an effort to provide strategies that direct the changes that take place in the population structures of a Fuzzy Non-homogeneous Markov System (F-NHMS) towards a desirable direction. Our goal is to maintain the population structure of the system, $\mathbf{N}(t)$, between two given population structures, $\mathbf{N}_1$ and $\mathbf{N}_2$, which is a very important issue in practical applications. More specifically, we study the aspect of attainability in a F-NHMS and give the input probability vector that achieves our aim. Maintainability is also studied by providing a necessary and sufficient condition such that $\mathbf{N}(t)$ lies between the two population structures, for each t. Finally, an illustrative example is provided.
Keywords: Markov systems, Fuzzy system models, Control theory.

1 Introduction - Problem statement

Let us first give a short description of a NHMS [Vassiliou, 1982]. Consider a population, which is stratified into classes according to different characteristics and let $S = \{1, 2, ..., n\}$ be the *set of states* of the system, which are assumed to be exclusive and exhaustive. Let also $\mathbf{N}(t) = [N_1(t), N_2(t), ..., N_n(t)]$ be the *expected population structure of the system* at time t, where $N_i(t)$ is the expected number of members in state i at time t. Let $T(t)$ denote the total number of members in the system and $\Delta T(t) = T(t+1) - T(t)$. Let us assume that the individual transitions between the states occur according to the sequence of matrices $\{\mathbf{P}(t)\}_{t=0}^{\infty}$ and that $\{\mathbf{p}_o(t)\}_{t=0}^{\infty}$ is *the sequence of input probability vectors.* Suppose, moreover, that the members that leave the system are transferred in a state $n+1$ denoting the external environment of the system and let $\{\mathbf{p}_{n+1}(t)\}_{t=0}^{\infty}$ be the *sequence of loss probability vectors.* Also assume that $q_{ij}(t) = p_{ij}(t) + p_{i,n+1}(t)p_{oj}(t)$, then we define the sequence of matrices

$\mathbf{Q}(t) = \mathbf{P}(t) + \mathbf{p}'_{n+1}(t)\mathbf{p}_o(t) = \{q_{ij}(t)\}_{i,j\in S}$, where $(\cdot)'$ denotes the transpose of the respective vector. $\{\mathbf{Q}(t)\}_{t=0}^{\infty}$ defines uniquely a non-homogeneous Markov chain, which is called the *embedded non-homogeneous Markov chain.* The (i,j) -element of $\mathbf{Q}(t)$ represents the total transition probability from state i to state j, in the time interval $(t-1,t]$. The expected number of members in the various states at time t is given by:

$$\mathbf{N}(t) = \mathbf{N}(t-1)\mathbf{Q}(t-1) + \Delta T(t-1)\mathbf{p}_o(t-1), or \tag{1}$$

$$\mathbf{N}(t) = \mathbf{N}(t-1)\mathbf{P}(t-1) + R(t-1)\mathbf{p}_o(t-1), \tag{2}$$

where $R(t)$ denotes the *expected number of new members in the system* at time t. In order to apply the model of a NHMS, $q_{ij}(t)$ (or $p_{ij}(t)$) and $p_{oi}(t)$ must be determined, $\forall\ i,j = 1,2,...,n$ and $\forall t$. This estimation obviously depends on statistical data analysis, it can be accomplished whenever enough data is provided and obviously introduces uncertainty due to measurement errors and lack of data. This is the main reason for considering fuzzy logic and fuzzy reasoning in Markov systems. In [Symeonaki *et al.*, 2000], [Symeonaki *et al.*, 2002] the concept of a F-NHMS was introduced. The asymptotic behaviour and variability of the system was provided, but this is only the initial step. We need to proceed in the opposite direction, since the projected structures will seldom coincide with what is desired. In this paper the goal is given and the objective is to provide the input probability vector that achieves the desired goal and the conditions under which the goal is maintained. More specifically, the objective here is to develop a useful methodology for obtaining the transition and input probabilities and provide thereafter the input probability vector such that the population structure of the system lies between two given population structures. A different approach to a similar end can be found in [Hartfiel, 1994]. In this paper the problem is expanded to population systems and more specifically to NHMS, where the transition, input and loss probabilities depend on time t. The present paper is organized as follows. In Section 2, a description of a F-NHMS is provided and the necessary parameters of the system are given. More specifically, attainability and maintainability in a F-NHMS is discussed. Section 3 provides an illustrative example of the conclusions of Section 2.

2 Input Control of a F-NHMS

In this section the central problem of input control of a F-NHMS with $S = \{1,2,...,n\}$ is discussed. It is assumed that the transition probability $p_{ij}(t)$ is a function of the population parameters (e.g. Longevity, Mortality, Fecundity, etc) of the system, i.e. $p_{ij}(t) = f_{ij}(pp_1, pp_2, ..., pp_l)$, where: $\sum_{j\in S} f_{ij}(pp_1, pp_2, ..., pp_l) \leq 1$, for any value of the population parameters $pp_1, pp_2, ..., pp_l$. The idea of the *population parameters* of the system was firstly presented in [Symeonaki *et al.*, 2000] and [Symeonaki *et al.*, 2002].

Each population parameter depends on the values of the basic parameters of the system. In order to determine the population parameters from the basic parameters of the system a Fuzzy Inference System (FIS) is used. The structure of a F-NHMS is illustrated in Figure 1.

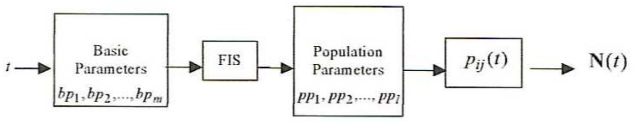

Fig. 1. The structure of the F-NHMS

Assume that the values of the $i-th$ basic parameter of the system range between two values α_i and b_i, i.e. the values of the $i-th$ basic parameter belong to the closed interval $[\alpha_i, b_i]$. A fuzzy partition $A^{(i)}$ of order d_i on the domain $[\alpha_i, b_i]$ is defined and a fuzzy partition $B^{(j)}$ of order r_j is also defined on the universe of discourse of the $j-th$ population parameter. The fuzzy partitions $A^{(i)}$ and $B^{(j)}$ are linguistic representations of their universe of discourses, therefore their elements are linguistic terms like "LOW", "HIGH", etc. The relationship of the crisp universe of discourses is represented using linguistic rules, that derive from the symbolic knowledge that the experts of the system possess and define a mapping of the fuzzy partitions $A^{(i)}$ to the fuzzy partitions $B^{(j)}$. This mapping is said to be a *fuzzy association* and represents the empirical, linguistic rules. As long as the elements of $A^{(i)}$ and the elements of $B^{(j)}$ have a linguistic meaning, heuristic or empirical linguistic rules can be used in order to describe the input-output relationship. We assume that all fuzzy partitions are complete [Stamou and Tzafestas, 1999]. The number of all different rules in the system is denoted by k and we can see that $k = d_1 d_2 \cdots d_m$. We denote by $w_i(t)$ the degree in which the rule i fires at time t. Each rule corresponds to a matrix $\mathbf{P}_i$ and it can easily be proved by induction that if we use as $t-$norm the *product*, then $\sum_{i=1}^{k} w_i(t) = 1$. Therefore, for each t, the transition matrix $\mathbf{P}(t)$ is of the form:

$$\mathbf{P}(t) = \sum_{i=1}^{k} w_i(t)\mathbf{P}_i, \tag{3}$$

with $\mathbf{P}_i\mathbf{1}' \leq \mathbf{1}'$ and $\sum_{i=1}^{k} w_i(t) = 1$, for each $t = 0, 1, 2, ...$, and $\mathbf{1}' = [1, 1, ..., 1]'$. Following the same reasoning for the sequence of input probability vectors, the vector $\mathbf{p}_o(t)$ is of the following form, for each t:

$$\mathbf{p}_o(t) = \sum_{i=1}^{m} u_i(t)\mathbf{p}_{o_i}, \tag{4}$$

$\mathbf{p}_{o_i}\mathbf{1}' = \mathbf{1}'$, $\sum_{i=1}^{m} u_i(t) = 1$, for each $t = 0, 1, 2, ...$, and $u_i(t)$ is the degree in which the rule i for the input probability vector $\mathbf{p}_o(t)$, fires. Therefore, from (2) the expected number of members in the various states of the system at time t, is given by:

$$\mathbf{N}(t) = \mathbf{N}(t-1)\sum_{i=1}^{k} w_i(t)\mathbf{P}_i + R(t-1)\sum_{i=1}^{m} u_i(t-1)\mathbf{p}_{o_i}, or \tag{5}$$

$$\mathbf{N}(t) = \mathbf{N}(0)\prod_{\tau=0}^{t-1}\sum_{i=1}^{k} w_i(\tau)\mathbf{P}_i + \sum_{\tau=1}^{t} R(\tau-1)\sum_{i=1}^{m} u_i(\tau-1)\mathbf{p}_{o_i}\prod_{j=\tau}^{t-1}\sum_{i=1}^{k} w_i(j)\mathbf{P}_i. \tag{6}$$

Let $M_{n,m}(F)$ define the set of all $n \times m$ matrices with elements from the field F.

Definition 1 *[Hartfiel, 1994]: Let two vectors $\boldsymbol{p}, \boldsymbol{q} \in M_{1,k}(R)$ for which it is $\boldsymbol{p} \leq \boldsymbol{q}$. The set of all vectors $\boldsymbol{x} \in M_{1,k}(R)$, which are such that $\boldsymbol{p} \leq \boldsymbol{x} \leq \boldsymbol{q}$, is called $box(\boldsymbol{p}, \boldsymbol{q})$, i.e. $box(\boldsymbol{p}, \boldsymbol{q}) = \{\boldsymbol{x} : \boldsymbol{p} \leq \boldsymbol{x} \leq \boldsymbol{q}\}$.*

Now let $\mathbf{N}_1, \mathbf{N}_2$ be two population structures such that $\mathbf{N}_1 \leq \mathbf{N}_2$. Then a NHMS is said to be *stably controllable* if we can maintain the population structure of the system between the desired structures $\mathbf{N}_1$ and $\mathbf{N}_2$ i.e. if: $\mathbf{N}_1 \leq \mathbf{N}(t) \leq \mathbf{N}_2$, $\forall t = 0, 1, 2, ...$. More specifically:

Definition 2 *[Symeonaki, 1998]: If $\forall t = 0, 1, 2, ...$ there exists an input vector $\boldsymbol{p}_o(t)$, such that for each $\boldsymbol{N}(t) \in box(\boldsymbol{N}_1, \boldsymbol{N}_2)$, there exists a $R(t)$ such that:*

$$\boldsymbol{N}(t)\boldsymbol{P}(t) + R(t)\boldsymbol{p}_o(t) \in box(\boldsymbol{N}_1, \boldsymbol{N}_2), \tag{7}$$

then the NHMS is called stably controllable.

Definition 3 *[Hartfiel, 1994]: A vector $\boldsymbol{x} \in M_{1,k}(R)$ is called $(\alpha - \boldsymbol{Q})-$ feasible, if $\boldsymbol{x}\boldsymbol{Q} \leq \alpha\boldsymbol{Q}, \alpha \in R_+$.*

Assume now that:

$$\mathbf{P}_{min} \leq \mathbf{P}(t) \leq \mathbf{P}_{max}, \forall t = 0, 1, 2, ..., \tag{8}$$

where: $\mathbf{P}_{min} = \sum_{i=1}^{k} w_{min_i}(t)\mathbf{P}_i$ and $\mathbf{P}_{max} = \sum_{i=1}^{k} w_{max_i}(t)\mathbf{P}_i$. Notice that this condition is not restrictive since in practice arbitrary movement would be highly undesirable if not impossible. Moreover, the condition applies to real applications where the exact transition probabilities cannot possibly be estimated. We assume now that $\mathbf{P}(t) = \mathbf{P}_{max}$, for some t and that the population structure $\mathbf{N}_1$ is $(1 - \mathbf{P}_{min})-$ feasible. The following theorem is now proved.

Theorem 1 *(attainability): Let a F-NHMS, which satisfies the above conditions. If:*

$$\boldsymbol{p}_o(t) = \frac{1}{R(t)}\boldsymbol{uB}$$

where:

$$\boldsymbol{u} = (\alpha_i), \forall i = 1, 2, ..., 2^k,$$

$$\boldsymbol{B} = [\boldsymbol{N}_1\boldsymbol{P}_{min}, \boldsymbol{N}_2\boldsymbol{P}_{max}] = [b_i], \forall i = 1, 2, ..., 2^k, b_i = X_i - Z_i,$$

and $\boldsymbol{N}_2$ is $(1 - \boldsymbol{P}_{max})-$ feasible, then $\boldsymbol{N}(t) \in box(\boldsymbol{N}_1, \boldsymbol{N}_2)$.

Proof. Let us assume that the structure $\mathbf{N}_2$ is $(1-\mathbf{P}_{max})-$ feasible. Therefore $\mathbf{N}_2\mathbf{P}_{max} \leq \mathbf{N}_2$. Moreover, from the hypothesis we have that $\mathbf{N}_1\mathbf{P}_{min} \leq \mathbf{N}_1$. Let $\mathbf{N}(t) \in box(\mathbf{N}_1, \mathbf{N}_2)$. Thus, from (8) we conclude that:

$$\mathbf{N}(t)\mathbf{P}(t) \in box(\mathbf{N}_1\mathbf{P}_{min}, \mathbf{N}_2\mathbf{P}_{max}). \tag{9}$$

Let X_i be the vertices of $box(\mathbf{N}_1, \mathbf{N}_2)$ and Z_i the vertices of $box(\mathbf{N}_1\mathbf{P}_{min}, \mathbf{N}_2\mathbf{P}_{max})$. It is assumed that the vertices are being numbered respectively, i.e.

$$(Z_i)_j = \begin{cases} (\mathbf{N}_1\mathbf{P}_{min})_j, \text{ iff } (X_i)_j = (\mathbf{N}_1)_j \\ (\mathbf{N}_2\mathbf{P}_{max})_j, \text{ iff } (X_i)_j = (\mathbf{N}_2)_j. \end{cases} \tag{10}$$

Given that $\mathbf{N}(t)\mathbf{P}(t) \in box(\mathbf{N}_1\mathbf{P}_{min}, \mathbf{N}_2\mathbf{P}_{max})$, the vector $\mathbf{N}(t)\mathbf{P}(t) \in box(\mathbf{N}_1\mathbf{P}_{min}, \mathbf{N}_2\mathbf{P}_{max})$ can be written as:

$$\mathbf{N}(t)\mathbf{P}(t) = \sum_{i=1}^{2^k} \alpha_i Z_i.$$

Let $\mathbf{u} = (\alpha_i)$ for $i = 1, 2, ..., 2^k$, $\mathbf{B} = [\mathbf{N}_1\mathbf{P}_{min}, \mathbf{N}_2\mathbf{P}_{max}, \mathbf{N}_1, \mathbf{N}_2] = [b_i]$ for $i = 1, 2, ..., 2^k$, where $b_i = X_i - Z_i$ and let s be the sum of the elements of the $(1 \times k)-$vector $\mathbf{uB}$. Therefore, if $\frac{1}{R(t)}\mathbf{uB}$, we have that:

$$\mathbf{N}(t)\mathbf{P}(t) + R(t)\mathbf{p}_o(t) = \sum_{i=1}^{2^k} \alpha_i Z_i + \sum_{i=1}^{2^k} \alpha_i (X_i - Z_i) = \sum_{i=1}^{2^k} \alpha_i (X_i) \tag{11}$$

i.e. $\mathbf{N}(t)\mathbf{P}(t) \in box(\mathbf{N}_1, \mathbf{N}_2)$. Therefore, $\mathbf{N}(t+1) \in box(\mathbf{N}_1, \mathbf{N}_2)$.

A necessary and sufficient condition that the system is stably controllable is given in the following theorem.

Theorem 2 *(maintainability): A F-NHMS is stably controllable if and only if the population structure $\boldsymbol{N}_2$ is $(1 - \boldsymbol{P}_{max})-$ feasible, where $\boldsymbol{P}_{max} = \sum_{i=1}^{k} w_{max_i}(t)\boldsymbol{P}_i$.*

Proof. Let us first assume that the system is stably controllable. Then, since: $\mathbf{N}_2 \in box(\mathbf{N}_1, \mathbf{N}_2)$ and $\mathbf{P}(t) = \mathbf{P}_{max}$, for some t, there exists an input vector $\mathbf{p}_o(t)$ and an $R(t)$ such that:

$$\mathbf{N}_2\mathbf{P}_{max} + R(t)\mathbf{p}_o(t) \in box(\mathbf{N}_1, \mathbf{N}_2),$$

i.e. $\mathbf{N}_2\mathbf{P}_{max} + R(t)\mathbf{p}_o(t) \leq \mathbf{N}_2$. Thus, $\mathbf{N}_2\mathbf{P}_{max} \leq \mathbf{N}_2$. Consequently, the structure $\mathbf{N}_2$ is $(1 - \mathbf{P}_{max})-$ feasible.

It is now assumed that the structure $\mathbf{N}_2$ is $(1 - \mathbf{P}_{max})-$ feasible. From Theorem 1 it follows that $\mathbf{N}(t+1) \in box(\mathbf{N}_1, \mathbf{N}_2)$. Therefore, the system is strongly controllable.

Putting the above results together, we conclude that in a F-NHMS if the structure $\mathbf{N}_2$ is $(1 - \mathbf{P}_{max})-$ feasible, then the limiting population structure given in [Symeonaki *et al.*, 2000] and [Symeonaki *et al.*, 2002] also lies between the two desired structures $\mathbf{N}_1$ and $\mathbf{N}_2$, i.e.:

$$\lim_{t\to\infty} \mathbf{N}(t) = \mathbf{N}(\infty) = T\mathbf{e}_i[\mathbf{I} - (\mathbf{I} - \sum_{i=1}^{s} v_i\mathbf{Q}_i)(\mathbf{I} - \sum_{i=1}^{s} v_i\mathbf{Q}_i)^\sharp] \in box(\mathbf{N}_1, \mathbf{N}_2),$$

where $(\cdot)^\sharp$ represents the generalized group inverse introduced in [Meyer, 1975], and $\mathbf{Q}_i = \mathbf{P}_j + \mathbf{p}'_{n+1_j}\mathbf{p}_{o_l}$ where j and l depend on i.

3 A numerical example

Let a NHMS with $S = \{1, 2, 3\}$ and let that a number of transition probabilities cannot be estimated due to lack of data. Suppose moreover that we have two factors that influence the transition probabilities. Furthermore, it is assumed that these population parameters depend upon two basic parameters. Combining the rules of the system with the generalized modus ponens (GMP) rule of inference [Klir and Yuan, 1995], [Stamou and Tzafestas, 1999] the multi-conditional approximate reasoning schema (system rules) is formulated. The system rule for the population parameter pp_1, for example, is described as follows:

1^{st} `RULE: IF` (x_1, x_2) `IS (SMALL, LITTLE), THEN` y_1 `IS LOW`
2^{nd} `RULE: IF` (x_1, x_2) `IS (SMALL, AVER), THEN` y_1 `IS LOW`
3^{rd} `RULE: IF` (x_1, x_2) `IS (SMALL, PLENTY), THEN` y_1 `IS LOW`
5^{th} `RULE: IF` (x_1, x_2) `IS (MED, AVER), THEN` y_1 `IS AVER`
6^{th} `RULE: IF` (x_1, x_2) `IS (MED, PLENTY), THEN` y_1 `IS HIGH`
7^{th} `RULE: IF` (x_1, x_2) `IS (LARGE, LITTLE), THEN` y_1 `IS AVER`
8^{th} `RULE: IF` (x_1, x_2) `IS (LARGE, AVER), THEN` y_1 `IS AVER`
9^{th} `RULE: IF` (x_1, x_2) `IS (LARGE, PLENTY), THEN` y_1 `IS HIGH`

Now let that: $\mathbf{P} = \begin{pmatrix} 0.6 & 0.1 & 0 \\ 0.1 & 0.5 & 0.1 \\ 0 & 0.1 & 0.5 \end{pmatrix}$ and $\mathbf{P}_{min} = \begin{pmatrix} 0.7 & 0.15 & 0.1 \\ 0.2 & 0.5 & 0.2 \\ 0 & 0.1 & 0.7 \end{pmatrix}$, and $\mathbf{P}_{min} \leq \mathbf{P}(t) \leq \mathbf{P}_{max}$, $\forall t = 0, 1, 2, ...$ Assume that we want to maintain the population structure of the system between the structures, $\mathbf{N}_1$ and $\mathbf{N}_2$, where: $\mathbf{N}_1 = (100\ 300\ 600)$, and $\mathbf{N}_2 = (100\ 350\ 650)$. $\mathbf{N}_1$ and $\mathbf{N}_2$ are $(1 - \mathbf{P}_{min})-$feasible and $(1 - \mathbf{P}_{max})-$feasible, respectively, since $\mathbf{N}_1\mathbf{P}_{min} = \begin{pmatrix} 90\ 220\ 330 \end{pmatrix} \leq \mathbf{N}_1$, $\mathbf{N}_2\mathbf{P}_{max} = \begin{pmatrix} 280\ 285\ 555 \end{pmatrix} \leq \mathbf{N}_2$. At time t, let:

$$\mathbf{N}(t)\mathbf{P}(t) = \begin{pmatrix} 185\ 261\ 442.5 \end{pmatrix} \in box(\mathbf{N}_1\mathbf{P}_{min}, \mathbf{N}_2\mathbf{P}_{max})$$

where $\mathbf{N}(t)\mathbf{P}(t) = \sum_{i=1}^{8} \alpha_i Z_i$, $\alpha_i = 0.125$, and X_i, Z_i are the vertices of $box(\mathbf{N}_1, \mathbf{N}_2)$ and $box(\mathbf{N}_1\mathbf{P}_{min}, \mathbf{N}_2\mathbf{P}_{max})$, respectively, as numbered in (10). Thus:

$$X_1 = \begin{pmatrix} 100\ 300\ 600 \end{pmatrix}, Z_1 = \begin{pmatrix} 90\ 220\ 330 \end{pmatrix}$$
$$X_2 = \begin{pmatrix} 100\ 300\ 650 \end{pmatrix}, Z_2 = \begin{pmatrix} 90\ 220\ 555 \end{pmatrix}$$
$$X_3 = \begin{pmatrix} 100\ 350\ 600 \end{pmatrix}, Z_3 = \begin{pmatrix} 90\ 285\ 330 \end{pmatrix}$$
$$X_4 = \begin{pmatrix} 100\ 350\ 650 \end{pmatrix}, Z_1 = \begin{pmatrix} 90\ 285\ 555 \end{pmatrix}$$
$$X_5 = \begin{pmatrix} 300\ 300\ 600 \end{pmatrix}, Z_5 = \begin{pmatrix} 280\ 220\ 555 \end{pmatrix}$$
$$X_6 = \begin{pmatrix} 100\ 300\ 650 \end{pmatrix}, Z_6 = \begin{pmatrix} 280\ 220\ 555 \end{pmatrix}$$
$$X_7 = \begin{pmatrix} 300\ 350\ 600 \end{pmatrix}, Z_7 = \begin{pmatrix} 280\ 285\ 330 \end{pmatrix}$$
$$X_8 = \begin{pmatrix} 300\ 350\ 650 \end{pmatrix}, Z_1 = \begin{pmatrix} 280\ 285\ 555 \end{pmatrix}$$

and matrix $\mathbf{B}^{'} = [b_i]^{'} = [X_i - Z_i]^{'}$ is

$$\mathbf{B}^{'} = \begin{pmatrix} 10 & 10 & 10 & 10 & 20 & 20 & 20 & 20 \\ 80 & 80 & 65 & 65 & 80 & 80 & 65 & 65 \\ 270 & 95 & 270 & 95 & 270 & 95 & 270 & 95 \end{pmatrix}$$

According to Theorems 1 and 2, if $R(t) = 291.875$ and $\mathbf{p}_o(t) = \begin{pmatrix} 0.052\ 0.284\ 0.7 \end{pmatrix}$, we have that: $\mathbf{N}(t+1) = \mathbf{N}(t)\mathbf{P}(t) + R(t)\mathbf{p}_o(t) = \begin{pmatrix} 200\ 333.5\ 646.875 \end{pmatrix} \in box(\mathbf{N}_1, \mathbf{N}_2)$ where $box(\mathbf{N}_1, \mathbf{N}_2)$ is shown in Figure 2.

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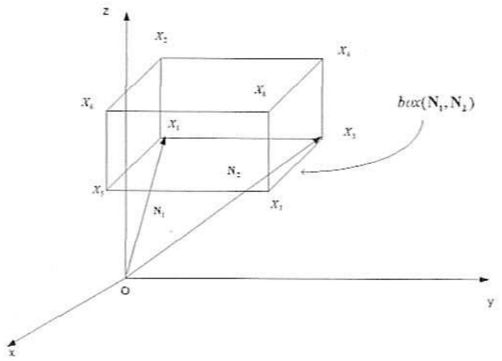

Fig. 2. $box(\mathbf{N}_1, \mathbf{N}_2)$

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