Chaotic Aspects of a GRM1 Innovation Diffusion Model

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Abstract. Chaotic behavior of a generalized rational (GRM1) innovation diffusion model is studied. The deterministic continuous version of this model was proposed, analyzed and applied in earlier publications. Here, the chaotic behavior is expressed through the discrete alternative of the continuous GRM1 model. The model shows symmetric and non-symmetric behavior expressed by a parameter σ. In this article it is found that when the diffusion parameter b and the parameter σ verify the relation $b/\sigma \geq 2$ then the chaotic aspects of the model appear. A method is proposed for fitting the model to the data. Time series data expressing the cumulative percentage of steel produced by the oxygen process in various countries are used. Characteristic graphs of the chaotic behavior are given and applications are presented.
Keywords: Chaotic modeling, Diffusion modeling, Speed of diffusion, Innovation diffusion, Non-linear models, Chaotic oscillations.

1 Introduction

It's become a commonplace to call this the information age, but an even more appropriate name might be the information age. In 1997, for example, the U.S. Patent and Trademark Office received 237.000 patent applications, a 15% increase from the year before. Also in 1997, the agency granted 124.127 patents, a record number and an increase of 16% from the volume it recorded at the beginning of the decade in 1991, a year that had also set a record for patent activity. At individual companies, the pace of innovation is even greater. In 1998, IBM Corp. received 2.657 patents for inventions, an increase of 54% from the number it won in 1997, according to a preliminary tally from the patent office. This was not a one-time surge, as IBM has been the leading recipient of U.S. patents for six consecutive years. And IBM was not alone in recording huge increases in U.S. patent activity last year: Sony Corp.'s patent number rose 53%, Eastman Kodak Co.'s 41%, and Motorola Inc.'s 33%, [Maguire and Hagen, 2001]. While not all patents translate into new products or new production methods, these figures clearly demonstrate a tendency, and this explosion of innovation activity presents significant challenges. One of the special challenges firms face in this decade is the challenge of designing, manufacturing, and distributing products in a global marketplace. If customers want new products, and they do, then companies have no choice but to gear up their processes to provide innovative features and the latest designs. This mean that companies must have a proper way to describe the competitive dynamics in a market, [Modis, 1997] and to predict how these new products or production methods will move in the marketplace. One of these methods is described in this paper. A model is proposed and some empirical data are explored. In earlier publications several innovation diffusion models where presented, analyzed and applied to real life data [Bass, 1969], [Mahajan and Schoeman, 1977], [Sharif and Kabir, 1976], [Skiadas, 1985], [Skiadas, 1986], [Skiadas, 1987], [Modis and Debecker, 1992]. A main direction of these applications was focused of the non-symmetric behavior of the models expressed by specific parameters. A relatively simple but very flexible model was proposed in an earlier publication based on a family of Generalized Rational Models, [Skiadas, 1985], [Skiadas, 1986], to express asymmetry during the innovation diffusion process. This model is expressed by the following differential equation:

$$\dot{f} = b\frac{f(F-f)}{F-(1-\sigma)f} \tag{1}$$

Where f is the number of adopters at time t, F is the total number of potential adopters, b is the diffusion parameter, and σ is a dimensionless parameter. This model has a point of infection varying from 0 to F when parameter σ decreases from ∞ to 0. Another interesting property of parameter σ is that it gives a measure of the asymmetry of the model. Perfect symmetry appear for $\sigma = 1$ when equation 1 reduces to equation 2 expressing the popular logistic model:

$$\dot{f} = bf\left(1-\frac{f}{F}\right) \tag{2}$$

For the last model it is easy to show that, by using the transformation:

$$\dot{f} = \frac{df}{dt} \approx \frac{\Delta f}{\Delta t} = \frac{f_{t+1}-f_t}{(t+1)-t} = f_{t+1}-f_t \tag{3}$$

it is expressed by the following difference equation:

$$f_{t+1} = f_t + bf_t\left(1-\frac{f_t}{F}\right) \tag{4}$$

Bifurcation and further chaotic behavior appear when $2 < b \leq 3$. Various applications of the logistic model in several disciplines showed that parameter b of the logistic model lies in very low limits lower that unity. Thus by using the logistic model it is not possible to express chaotic behavior in real situations as the estimated values of parameter b fail to reach the limit at which chaotic behavior appear. On the other hand provided data for various cases show that oscillations and chaotic behavior appear quite frequently and especially when the diffusion process is close to the upper limit F. Moreover when the logistic model is applied in the form:

$$X_{t+1} = bX_t(1 - X_t) \tag{5}$$

Where $X_t = f_t/F$ then, bifurcation and chaotic behavior appears when $3 < b \leq 4$.

Chaos appears for very high values of b, which are not reasonable for real situations. Clearly the model form (4) is more correct as a discrete logistic model expressing behavior similar to the continuous model resulting from differential equation 2. This can be found in an older application done by [Nash, 1976]. The aim of this paper is to show that the model (1) exhibits chaotic behavior for values of parameter b that are quite low and are in accordance to the values estimated in real situations. This is achieved by the help of the flexible parameter σ, which gives a measure of the asymmetry of the model. The chaotic behavior of the model is analyzed and illustrated by using significant graphs. Finally, real life applications are presented.

2 The Generalized Rational Model

The model proposed is a discrete version of the continuous one expressed by equation 1. By introducing the approximation of $\dot{f}$ from equation 3 in the differential equation 1 the following difference equation results:

$$f_{t+1} = f_t + b\frac{f_t(F - f_t)}{F - (1 - \sigma)f_t} \tag{6}$$

Some interesting properties of this model are illustrated in Figures 1to 3.

In Figure 1a the proposed model shows the classical sigmoid form, whereas in Figure 1b the bifurcation appear as a simple oscillation. In Figure 1c a more complicated oscillation with four distinct oscillating levels appears, whereas in Figure 1d - 1f a total chaotic form appears. In all cases presented here the starting value is $f_0 = 1$, the upper limit $F = 100$, $b = 0.3$ and σ takes various values. The value selected for b is within the range 0.1 to 0.5, which is valid in real situations. By varying the dimensionless parameter σ several forms of the model appear.

A very important point is the estimation of the values of parameters b and σ for which bifurcation appear. The presence of the first oscillations and the onset to chaos, which follows, is a very important point when studying innovation diffusion systems. According to the theory of chaotic models, bifurcation for the model (6) starts when:

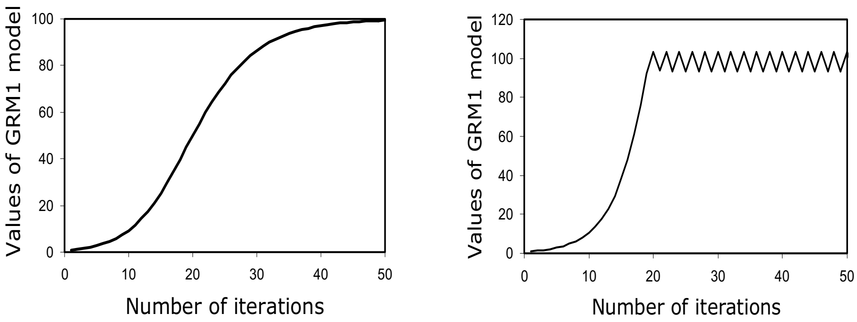

Fig. 1. GRM1 model for a) $b = 0.3$ and $\sigma = 2$ and b) $b = 0.3$ and $\sigma = 0.13$

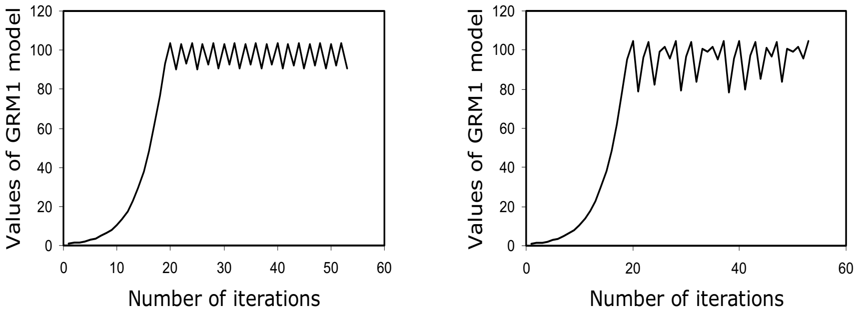

Fig. 2. GRM1 model for a) $b = 0.3$ and $\sigma = 0.12$ and b) $b = 0.3$ and $\sigma = 0.10$

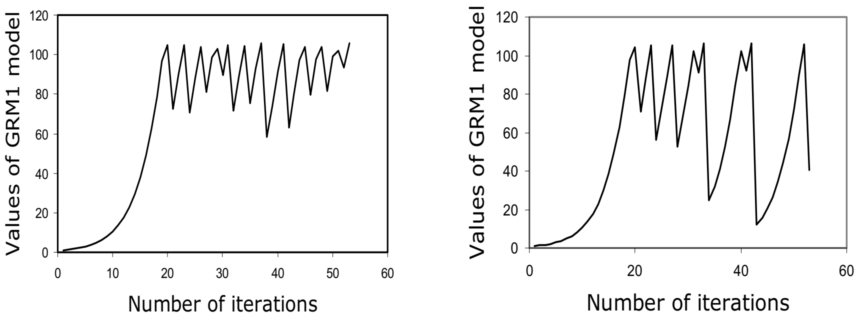

Fig. 3. GRM1 model for a) $b = 0.3$ and $\sigma = 0.09$ and b) $b = 0.3$ and $\sigma = 0.08$

$$f'_{t+1} = -1, \quad f_{t+1} = f_t \tag{7}$$

By applying equations 7 to equation 6 results the following relation for parameters b and σ:

$$\frac{b}{\sigma} = 2 \tag{8}$$

When $b/\sigma > 2$ then oscillation and chaotic behavior appear by gradually augmenting the fraction b/σ . When $\sigma = 1$ which is the case for the logistic model bifurcation appear for values of $b > 2$.

It is also possible to obtain analytic form for the values of f_t after the first bifurcation point and before the second. To achieve this we consider that $f_{t+2} = f_t$. The exact formula is given by:

$$f_t = F\frac{(b+2) \pm \sqrt{\frac{b(b+2)(b-2\sigma)}{(b-2\sigma+2)}}}{2(b+\sigma-1)} \tag{9}$$

For the logistic model $\sigma = 1$ and thus equation 9 reduces to:

$$f_t = F\frac{(b+2) \pm \sqrt{(b+2)(b-2)}}{2b} \tag{10}$$

When $b > 2\sigma$ in equation 9 or $b > 2$ in equation 10 the system oscillates at the values of f_t given by the above formulas respectively. When b is higher of the values expressing the second bifurcation point four distinct oscillating levels appear and later eight and finally 2^n points. For sufficient specifically high values of b, n is very high and the system exhibits chaotic oscillations.

3 Parameters' Estimation of GRM1 Model

The parameters of the discrete GRM1 model are estimated by an Iterative non-linear regression analysis algorithm by minimizing the sum of squared errors ($S = SSE$):

$$S = \sum \epsilon_t^2 = \sum_{t=1}^{n} (y_t - f_t)^2 \tag{11}$$

where ϵ_t is the error term of the stochastic equation:

$$y_t = f_t + \sum_{i=1}^{n} \frac{\vartheta f_t}{\vartheta a_i} \Delta a_i + \epsilon_t \tag{12}$$

y_t denotes provided data and f_t is calculated for every t from equation 6, given a set of initial values of parameters a_i. The estimation of parameters is highly sensitive in the presence of oscillations and chaotic oscillations in the provided data. For a better fitting it was decided to use the non-linear estimation method proposed by Nash for the discrete Logistic model for only three parameters of the model and retaining the dimensionless parameter σ. This parameter is gradualy changed as the iterative procedure proceeds until the sum of squared errors is minimized. The starting values of the partial derivatives need the estimation of the following forms given a set of initial values for the parameters of the model:

$$\frac{\vartheta f_1}{\vartheta b} = \frac{f_0(F - f_0)}{F - (1-\sigma)f_0} \tag{13}$$

$$\frac{\vartheta f_1}{\vartheta f_0} = 1 + b\frac{F^2 - 2Ff_0 + (1-\sigma)f_0^2}{(F - (1-\sigma)f_0)^2} \tag{14}$$

$$\frac{\vartheta f_1}{\vartheta F} = b\sigma\left(\frac{f_0}{F}\right)^2 \tag{15}$$

After the above estimation of the initial values of the parial derivatives the iterative procedure continues the estimation by using the following formulae:

$$\frac{\vartheta f_{t+1}}{\vartheta b} = \frac{\vartheta f_t}{\vartheta b}(1 + bk_t) + \frac{f_t(F - f_t)}{F - (1-\sigma)f_t} \tag{16}$$

$$\frac{\vartheta f_{t+1}}{\vartheta f_0} = \frac{\vartheta f_t}{\vartheta f_0}(1 + bk_t) \tag{17}$$

$$\frac{\vartheta f_{t+1}}{\vartheta F} = \frac{\vartheta f_t}{\vartheta F}(1 + bk_t) + \frac{b\sigma f_t^2}{(F - (1-\sigma)f_t)^2} \tag{18}$$

where:

$$k_t = \left(F^2 - 2Ff_t + \frac{(1-\sigma)f_t}{F - (1-\sigma)f_t}\right)^2 \tag{19}$$

4 Illustrations

Time series data expressing the cumulative percentage of steel produced by the oxygen process in various countries are used from an earlier application, [Poznanski, 1983]. Figure 4 illustrates the diffusion of Oxygen steel technology in Spain from 1968 to 1980, for a number of 13 years. The actial data include 18 years but, it is more appropriate to study the last part of the time series data as this part shows the characteristic oscillations that are of special interest in this study. The small cycles indicate the actual data, the dotted line chracterizes the path of the logistic model and the simple line is for the GRM1 model.

Parameter estimates and the sum of squared errors are summarized in Table 1. The parameter b for the Logistic model is relatively high but is far

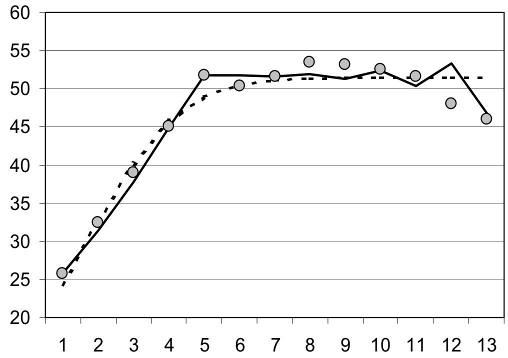

Fig. 4. Spain, Oxygen Steel Process (1968-1980)

Model	b	l	F	$\sigma(b/\sigma)$	SSE
Logistic	0.6309	24.373	51.474	-	72.838
GRM1	0.2331	25.779	51.736	0.084 (2.775)	41.748

Table 1. Parameter Estimates and Sum of Squared Errors (SSE) for Logistic and GRM1 Models in Spain from 1968 to 1980

away from the value needed for the start of bifurcation ($b = 2$). The form of the logistic path presented in the Figure 4 has a smouth form. The model fail to express the oscillating behavior of the actual case studied. Instead the GRM1 model shows a value for the parameter b lower to that of the Logistic model but the extra parameter σ accounts for the presence of oscillating and further of chaotic behavior as the fraction $b/\sigma = 2.775 > 2$. The estimated values for the parameters l and F are very close for both models. The ability of GRM1 model to follow the oscillating behavior of actual data is illustrated in the above Figure and is also expressed by the strong improvement of the Sum of Squared Errors (SSE).

Figure 5 illustrates the diffusion of oxygen steel technology in Italy from 1970 to 1980. The process ends in an oscillating form. The discrete Logistic fails to express these oscillations whereas the discrete GRM1 shows a considerable flexibility to approximate the real data. The sum of the squared errors is very low in the case of GRM1 model compared to that of the Logistic as is demonstrated in Table 2. The fraction $b/\sigma = 3.5292$ for the GRM1 model accounts for the chaotic behavior.

The actual data for the diffusion of the oxygen steel process in Luxemburg are of considerable interest as they cover the scale from 1.5 % during 1962 to that of 100 % in 1980 (Figure 6). The GRM1 model showed a good flexibility as it covers the fast growth process in the first stages of the diffusion process followed by a sudden turn to the high platform of 100%. The small also

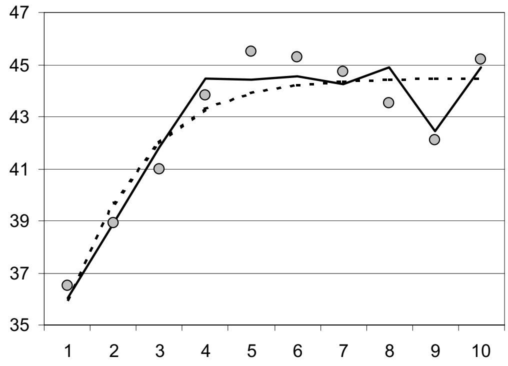

Fig. 5. Italy Oxygen Steel Process (1970-1980)

Model	b	l	F	$\sigma(b/\sigma)$	SSE
Logistic	0.5447	35.957	44.473	-	15.330
GRM1	0.08823	36.0402	44.4614	0.025 (3.5292)	7.431

Table 2. Parameter Estimates and Sum of Squared Errors (SSE) for Logistic and GRM1 Models in Italy from 1970 to 1980

flictuations at the end of the process are also simulated quite well as the fraction $b/\sigma = 3.609$ accounts for the chaotic region of the model. Figure 6 illustrates the case of Luxemburg for the following estimated values for the parameters: $b = 0.1931$, $l = 7.968$, $F = 99.669$ and $\sigma = 0.0535$. The mean squared error is $MSE = 20.872$.

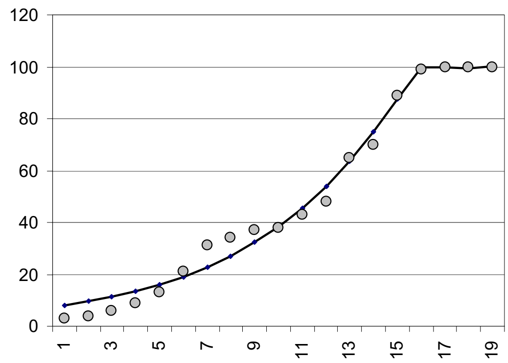

Fig. 6. Luxemburg Oxygen Steel Process (1962-1980)

The flexibility and the ability of GRM1 model to simulate growth processes that show at the end of the process oscillations and also chaotic oscillations is demonstrated in the following case of the diffusion of oxygen steel technology in Bulgaria from 1968 to 1978 (see Figure 7). The estimated parameters have values $b = 0.04046$, $l = 49.2425$, $F = 58.412$ and $\sigma = 0.012$. The sum of squared errors is $SSE = 21.431$ and the fraction $b/\sigma = 3.3718$ indicates that the model behave in the chaotic region.

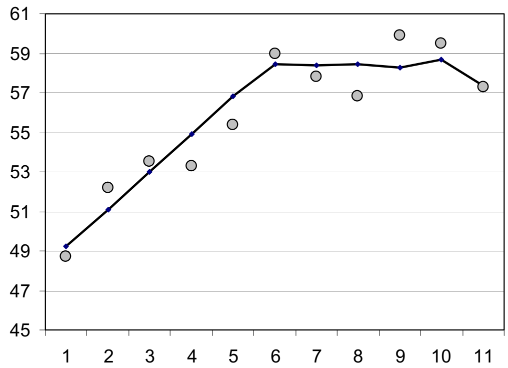

Fig. 7. Bulgaria Oxygen Steel Process (1968-1978)

5 Summary and Conclusions

A nonsymmetric innovation diffusion model is presented and analyzed regarding the chaotic behavior. It is shown that this model exhibits bifurcation and further chaotic behavior for some values of the fraction b/σ of the parameters b and σ. Real time-series data are used and parameters are estimated by an Iterative non-linear algorithm showed that in some cases the model performs oscillations (the fraction b/σ has values higher than 2) whereas in other cases the model showed the classical sigmoid form.

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