A reliability system governed by a LDQBD process

Pérez-Ocón, Rafael[1], Montoro-Cazorla, Delia[2], and Ruiz-Castro, Juan Eloy[1]

[1] Departamento de Estadística e I.O. Universidad de Granada. Spain
(e-mail: rperezo@ugr.es, jeloy@ugr.es ; http://www.ugr.es/ sim jeloy)
[2] Departamento de Estadística e I.O. Universidad de Jaén. Spain
(e-mail: dmontoro@ujaen.es)

Abstract. We study an n-unit system. The system functions as long as there is one unit online and the others in warm standby. When a unit fails it goes to repair. There is a repairman. The units are repaired following the arrival order. The operational and repair times follow phase-type distributions. The warm-standby units have a lifetime exponentially distributed. We construct the Markov model that govers the system and calculate performance measures. The mathematical expressions are algorithmically and computationally implemented, using the Matlab programme.
Keywords: Reliability, Availiability, Markov process, Rate of occurrence of failures (Rocof), Level-Dependent-Quasi-Birth-and-Death process.

1 Introduction

The literature on reliability systems concerning with Markov processes is related to systems with units having exponentially distributed lifetimes or extensions of it, such as Erlang, generalized Erlang or hyperexponential. It is known that the phase-type distributions (PH-distributions) constitute a large class that contains all the previous ones. This class has been studied in detail by [Neuts, 1981] and it has been recently applied in reliability by [Pérez-Ocón and Montoro-Cazorla, 2004a], [Neuts *et al.*, 2000], [Pérez-Ocón and Montoro-Cazorla 2004b].

When PH-distributions are involved in the modelization of systems, the generator of the Markov model that governs the system in certain finite cases has a tri-diagonal block structure, which characterizes the classes of quasi-birth-and-death processes (QBD processes) and level-dependent quasi-birth-and-death processes (LDQBD processes).These processes have been studied in [Latouche and Ramaswami, 1999] and oftenly considered in queueing theory ([Bright and Taylor, 1997], [Naoumov, 1997] and references therein). However, we have no information concerning the application of these processes in reliability theory. Recently, a multiple cold standby system involving PH distributions and governed by a QBD process has been studied by [Pérez-Ocón and Montoro-Cazorla, 2004b] . In the present paper we extend that work considering the system in warm standby, being the lifetime of the units in standby exponentially distributed. The stochastic process that governs the system results then a LDQBD process.

For this system, the stationary probability vector, the availability, and the rate of occurrence of failures are calculated. In addition, the distributions of the up and down periods are determined. The steady-state probability vector is calculated following the general methodology provided by [Naoumov, 1997] for solving linear systems with tri-diagonal block matrices. A numerical example is presented

We summarize the following definitions used in the paper.

Definition 1 *The distribution $H(\cdot)$ on $[0,\infty[$ is a phase-type distribution (PH-distribution) with representation (α, T), if it is the distribution of the time until absorption in a Markov process on the states $\{1,\ldots,m,m+1\}$ with generator by blocks*

$$\begin{pmatrix} T & T^0 \\ 0 & 0 \end{pmatrix}$$

and initial probability vector (α, α_{m+1}),where α is a row m-vector. We assume that the states $\{1,\ldots,m\}$ are all transient and m+1 absorbent. The distribution $H(\cdot)$ is given by

$$H(x) = 1 - \alpha \exp(Tx)e, \qquad x \geq 0.$$

It will be denoted that $H(\cdot)$ follows a $PH(\alpha, T)$ distribution.

Definition 2 *A level-dependent quasi-birth-and-death process (LDQBD process) on the state space $E = \{(i,j), 0 \leq i \leq n, 1 \leq j \leq m\}$, is a Markov process the infinitesimal generator of which is given by*

$$Q = \begin{pmatrix} B_{0,0} & B_{0,1} & & & & \\ B_{1,0} & A_1^1 & A_0^1 & & & \\ & A_2^2 & A_1^2 & A_0^2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & A_2^{n-1} & A_1^{n-1} & B_{n-1,n} \\ & & & & B_{n,n-1} & B_{n,n} \end{pmatrix} \tag{1}$$

The general definition of these type of processes can be modified depending on the boundary behavior. If we put $A_0^k = A_0, A_1^k = A_1, A_2^k = A_2$, $k = 1, 2, \ldots, n-1$, we get a QBD process.

Definition 3 *If A and B are rectangular matrices of dimensions $m_1 \times m_2$ and $n_1 \times n_2$ respectively, their Kronecker product $A \otimes B$ is the matrix of dimensions $m_1 n_1 \times m_2 n_2$, written in compact form as $(a_{ij}B)$.*

2 The model

Let us consider a repairable n-system, with one unit online and the rest in one of the following three situations: in warm standby, being repaired or waiting for repair. There is one repairman, which serves following the arrival order of the units. The unit online has a lifetime distributed as a $PH(\alpha, T)$ with m operational phases. The units in warm standby have lifetime distributed following $\exp(\lambda_s)$. The repair time follows a distribution $PH(\beta, S)$ with k repairing phases. The repair is as good as new. These times are independent. If there is a unit online and a repair is completed, it goes to standby. When all the units are non-operational and a repair is completed, the repaired unit becomes the unit online.

For introducing a Markov model it is necessary to identify exponentially distributed states in the evolution of the system. These will be the operational and repair phases. Thus, the states will be triplets indicating theses phases and the number of non-operational units. The state spaces is given by $S = S_1 \cup S_2 \cup S_3$,with

$$\begin{aligned}
S_1 &= \{(0, j), 1 \le j \le m\},\\
S_2 &= \{(i, j, l), 1 \le i \le n-1, 1 \le j \le m, 1 \le l \le k\},\\
S_3 &= \{(n, l), 1 \le l \le k\},
\end{aligned}$$

where i denotes the number of non-operational units, j the operational phase of the online unit, and l the repair phase of the unit under repair. The system macro-states are given in the set $S = \{i,\, i = 0, 1, \ldots, n\}$.

The infinitesimal generator, Q, is calculated from the transition rates among the macro-states. This generator is composed of blocks and the matrix is like the one given in (1), with the blocks in (2).

In the expressions below, the matrix I denotes the identity matrix of appropriate order.

$$\begin{aligned}
B_{0,0} &= T - (n-1)\lambda_s I,\\
B_{0,1} &= \left[T^0\alpha + (n-1)\lambda_s I\right] \otimes \beta,\\
B_{1,0} &= I \otimes S^0,\\
A_2 &= I \otimes S^0\beta,\\
A_1^{(i)} &= [T \oplus S] - (n-i-1)\lambda_s I, \quad i = 1, 2, \ldots, n-1\\
A_0^{(i)} &= \left[T^0\alpha \otimes I\right] + (n-i-1)\lambda_s I, \quad i = 1, 2, \ldots, n-2\\
B_{n-1,n} &= T^0 \otimes I,\\
B_{n,n-1} &= S^0\alpha \otimes \beta,\\
B_{n,n} &= S.
\end{aligned} \tag{2}$$

3 Stationary probability vector

We use $\pi = (\pi_0, \pi_1, \ldots, \pi_{n-1}, \pi_n)$ to denote the stationary-probability vector, which satisfies the matricial equation $\pi Q = 0$, subject to the normalization condition $\pi e = 1$.

To solve resulting system we use previous results ([Naoumov, 1997], Proposition 18). It is obtained that the stationary vector can be recursively obtained in terms of π_0 and rate matrices as $\pi_j = \pi_0 \prod_{k=0}^{j-1} R_k,\ j = 1, \ldots, n,$ being

$$\begin{aligned}
R_{n-1} &= -B_{n-1,n}B_{n,n}^{-1}, \\
R_{n-2} &= -A_0^{(n-2)}(A_1^{(n-1)} + R_{n-1}B_{n,n-1})^{-1}, \\
R_{j-1} &= -A_0^{(j-1)}\left(A_1^{(j)} + R_jA_2\right)^{-1}, \quad j = n-2, \ldots, 2 \\
R_0 &= -B_{0,1}(A_1^{(1)} + R_1A_2)^{-1}
\end{aligned}$$

The vector π_0 is determined by the equation $\pi_0(B_{0,0} + R_0B_{1,0}) = \mathbf{0}$ subjected to the normalization condition $\pi_0\left(\sum_{j=0}^{n}\prod_{i=0}^{j-1} R_i\right)e = 1$.

4 Performance measures

The performance measures will be given by means of the stationary probability vector and, consequently, from the matrices R. Below two of these measures appear, the availability and several rates of occurrence of failures: for the unit online and for the system.

The availability of the system is the probability that the system will be operational, thus:

$$A = \sum_{i=0}^{n-1} \pi_i e = \pi_0 \left(\sum_{i=0}^{n-1}\prod_{k=0}^{i-1} R_k\right) e = 1 - \pi_n e.$$

We now calculate the rate of occurrence of failures for the unit online, whose expression results:

$$v_1 = \pi_0 T^0 + \pi_0 \left(\sum_{i=1}^{n-1}\prod_{k=0}^{i-1} R_k\right)(T^0 \otimes e).$$

The mean number of times that the system is down per unit time.is given by

$$\nu_2 = \pi_{n-1}(T^0 \otimes e) = \pi_0 \left(\prod_{k=1}^{n-2} R_k\right)(T^0 \otimes e).$$

5 Distributions of the up and down periods

It is useful to know the distribution of the times during the system is operational or is being repaired in the long run. These are of special importance in systems that require a high reliability. We will show that these random times follow PH-distributions. In the references we have found different ways to define an up period. One is the timespan between the point at which all the units are initially operational (macro-state 0) and the point at which all the units are not operational by first time (macro-state n). Another definition is the timespan between the instant in which an unit completes its repair while the others are non-operational (the system enters the macro-state $n-1$ from n) and the instant in which for the first time the system is non-operational (enters the macro-state n). For calculating the distribution function of this time, we consider a modified Markov process from the original, with the same operational macro-states and identifying the non-operational macro-states in a new absorbent macro-state that will be denoted by n^*. The up period is the time up to the absorption by the macro-state n^*, and thus the distribution will be a PH-distribution. The generator Q^* of this new Markov process is derived from the expression (1) where the block $B_{n,n-1}$ is a null row vector, $B_{n,n} = 0$ and $B_{n-1,n}$ is replaced by the column vector $B_{n-1,n}e$.

The representation of the up period is (γ_u,L_u), matrix L_u being the one calculated from Q^* eliminating the row and the column corresponding to the macro-state n^*. The initial conditions need to be chosen so as to reflect the physical conditions of the system at time $t = 0$. If all units are operational at this point, the initial vector can be chosen as $(\alpha, 0, \ldots, 0)$. Choosing this definition we focus on the initial warranty period of the system, that is, the time to system failure given that initially all the units are operational. However, if we consider the second definition of the up period given above, the initial vector can be chosen as $(0, \ldots, 0, \alpha \otimes \beta)$. It is possible to express the initial condition in terms of the stationary probability vector, then, the initial vector considering the first definition above can be chosen as

$$\gamma_u = \left[\frac{\pi_0}{\pi_0 e}, \mathbf{0}\right].$$

The operational mean time is

$$MTTF = -\gamma_u L_u^{-1} e.$$

The down period begins when the only operational unit fails (the rest are in repair or waiting for repairing), and finishes at the point when the first repair is completed. This period follows a $PH(\gamma_d, S)$, where γ_d is determined as follows. Let $\gamma_d(l)$ $1 \leq l \leq k$, be the stationary probability that the unit under repairing occupies the phase l. The system initiates its down-period in the infinitesimal interval $(t, t + dt)$ with probability $\pi_n(T^0 \otimes e)dt$, then,

	$n = 10$		
π_0	0.0004	*	*
π_n	0.0463	0.0142	0.0115

	$n = 50$		
π_0	*	*	*
π_n	0.0470	0.0144	0.0117

	$n = 20$		
π_0	*	*	*
π_n	0.0468	0.0144	0.0117

	$n = 100$		
π_0	*	*	*
π_n	0.0471	0.0144	0.0117

Table 1. Stationary probabilities π_0, π_n for different values of n

$$\gamma_d(l) = \frac{\sum_j \pi_{n-1}(j,l)T_j^0}{\pi_{n-1}(T^0 \otimes e)}, \quad 1 \leq l \leq k,$$

being $\pi_{n-1}(j,l)$ the probability that at any time $n-1$ components of the system are down with the unit online in phase j and the unit under repair in phase l. The initial vector yields then

$$\gamma_d = (\gamma_d(l))_{1 \leq l \leq k}.$$

The mean time that the system remains down is given by

$$MTTD = -\gamma_d S^{-1} e.$$

6 Numerical application

In this section we apply the calculations performed above to a practical case, preserving the notation of the previous ones. We consider the following representations for the PH-distributions of the operational and repair times.

$$\alpha = (1,0,0) \qquad\qquad \beta = (1,0,0)$$

$$\mathbf{T} = \begin{pmatrix} -0.0027 & 0.0027 & 0 \\ 0 & -0.008 & 0.008 \\ 0 & 0 & -0.02878 \end{pmatrix}, \mathbf{S} = \begin{pmatrix} -0.02 & 0.02 & 0 \\ 0.01 & -0.08 & 0.07 \\ 0.005 & 0 & -0.1 \end{pmatrix}.$$

Let us study the behavior of the system defined in Section 2 with these numerical values for different number of units n. The stability of the measures in terms of the number of units is calculated. The failure rate for the units in standby will be $\lambda_s = 0.03$.

In Table 1 we present the values of π_0 and π_n for different values of n, showing that for n close to 20 the probabilities remain stable when n increases. The values of π_0 are very close to $\mathbf{0}$ for $n \geq 20$, and the ones corresponding to π_n tends to (0.0470,0.0144,0.0117) when n increases.

These values indicate that there are frequently non-operational units, and the system is down with a probability close to 7.31% when there a few units.

n	A	v_1	v_2	$MTTF$	ρ	L
10	0.928	0.002	0.001	828.193	0.999	8.659
20	0.927	0.002	0.001	842.439	1	18.534
30	0.927	0.002	0.001	851.241	1	28.402
50	0.927	0.002	0.001	862.998	1	48.128

Table 2. Performance measures for different number of units

In Table 2 we present the performance measures that have been introduced in previous sections. We use ρ to denote the utilization factor, that is, the proportion of time that the repairman is busy, and L denotes the mean number of non-operational units. MTTF is the mean time of the up period.

The availability decreases slightly when the number of units increases, and stabilizes at around the 92.7%. The different rate of occurrence of failures change slowly with n. The mean number of units failing per unit time increases softly; for example, for a system with 30 units, the mean time between two consecutive unit failures is about 75.757 t.u. The utilization factor of the repairman is very near to 1, so that the repair system is almost saturated, and the mean number of units in the repair channel consequently increases. The mean number of total failure per unit time of the system is 0.001.

Final note. Taking $\lambda_s = 0$ in our model we have an 1-out-of-n-system, where one unit is online and the others in cold standby. Thus, the stochastic process that governs the system is a quasi-birth-and-death process (QBD process), that has been studied in [Pérez-Ocón and Montoro-Cazorla, 2004a].

Acknowledgement. This research was supported by the Proyecto MTM 2004-03672 from the Ministerio de Ciencia y Tecnologia, Spain.

References

[Bright and Taylor, 1997]L. Bright and P.G. Taylor. Equilibrium distributions for level-dependent quasi-birth-and-death processes. *In S.R. Chakravarty and A.S. Alfa,Eds., Matrix-analytic methods in stochastic models*, pages 359–375, 1997.

[Latouche and Ramaswami, 1999]G. Latouche and V. Ramaswami. *Introduction to Matrix Analytic Methods in Stochastic Modeling.* ASA-SIAM, 1999.

[Naoumov, 1997]V. Naoumov. Matrix-multiplicative approach to quasi-birth-and-death processes analysis. *In S.R. Chakravarty and A.S. Alfa, Eds., Matrix-analytic methods in stochastic models*, pages 87–106, 1997.

[Neuts *et al.*, 2000]M.F. Neuts, R. Pérez-Ocón, and I. Torres-Castro. Repairable models with operating and repair times governed by phase type distributions. *Advances in Applied Probability*, pages 468–479, 2000.

[Neuts, 1981]M.F. Neuts. *Matrix Geometric Solutions in Stochastic Models. An Algorithmic Approach.* John Hopkins, Univ. Press., 1981.

[Pérez-Ocón and Montoro-Cazorla, 2004a]R. Pérez-Ocón and D. Montoro-Cazorla. A multiple system governed by a quasi-birth-and-death process. *Reliability Engineering & System Safety*, pages 187–196, 2004.

[Pérez-Ocón and Montoro-Cazorla, 2004b]R. Pérez-Ocón and D. Montoro-Cazorla. Transient analysis of a repairable system using phase type distributions and geometrics processes. *IEEE Transactions in Reliability*, pages 185–192, 2004.