On Two New Methods for Constructing Multivariate Probability Distributions with System Reliability Motivations

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Abstract. Two distinct methods of construction of some interesting new classes of multivariate probability densities are described and applied. As common result of both procedures, two n-variate pdf classes are obtained. The classes are considered as multivariate generalizations of the classes of univariate Weibullian and gamma pdfs. Example of an application of the obtained n-variate pdfs to the problem of modeling the reliability of multicomponent systems with stochastically dependent life-times of their components is given. Possibility to construct an extension of the considered random vectors to stochastic processes is communicated. Application of the so obtained (ex-Weibullian) stochastic processes as highly non-Markovian but simple models for maintenance of systems, with a history of all past repairs recorded, is presented.
Keywords: multivariate probability density, system reliability and maintenance modeling, highly non-Markovian models, n-variate ex-exponential, ex-Weibullian, ex-gamma pdfs, pseudoaffine transformations on $\mathbf{R}^n$.

1 On pseudoaffine transformations

Suppose $T_1, T_2, \ldots, T_n$ are independent random variables and for each $i = 1, \ldots, n$, T_i has a pdf that belongs to one of the following four classes of probability distributions: Gaussian, exponential, Weibullian or gamma (i.e., all T_i 's are assumed to be in exactly one of the above classes). To any so defined random vector $(T_1, T_2, \ldots, T_n)$ apply a member from the following new class of $\mathbf{R}^n \to \mathbf{R}^n$ pseudoaffine transformations (see [Filus and Filus, 2001b], [Filus and Filus, 2003b]) defined by the following scheme (recall that the well known ordinary affine transformations in $\mathbf{R}^n$ are usually understood to be compositions of nonsingular linears and translations on $\mathbf{R}^n$):

$$
\begin{aligned}
X_1 &\stackrel{d}{=} \phi_0 T_1 + \psi_0, \\
X_2 &\stackrel{d}{=} \phi_1(X_1) T_2 + \psi_1(X_1), \\
&\cdots\cdots\cdots \\
X_n &\stackrel{d}{=} \phi_{n-1}(X_1, \ldots, X_{n-1}) T_n + \psi_{n-1}(X_1, \ldots, X_{n-1}),
\end{aligned}
\tag{1}
$$

where ϕ_0, ψ_0 are constants, with $\phi_0 \neq 0$, and the functions

$$\phi_1(x_1), \ldots, \phi_{n-1}(x_1, \ldots x_{n-1}), \psi_1(x_1), \ldots, \psi_{n-1}(x_1, \ldots, x_{n-1}),$$

called parameter functions, are assumed to be continuous at least with respect to each of their arguments $x_1, \ldots, x_{n-1}$ separately, whenever present. It is also assumed that, for $j = 1, \ldots, n-1$, $\phi_j(x_1, \ldots, x_j) \neq 0$. In general, especially in reliability applications of the models to appear in this text, both the following conditions: $\phi_j(0, \ldots, 0) = 1$, and $\psi_j(0, \ldots, 0) = 0$ should hold too. If $\psi_0 = \psi_1(x_1) = \psi_2(x_1, x_2) = \ldots = \psi_{n-1}(x_1, \ldots, x_{n-1}) \equiv 0$, then the scheme (1) reduces to the pattern that will be called '(diagonal) pseudolinear' as it is a generalization of linear mappings in $\mathbf{R}^n$. As it can easily be shown all the transformations (1) are easily reversible and the jacobians of their inverses have remarkably simple product form:

$$\partial(t_1, \ldots, t_n)/\partial(x_1, \ldots, x_n) = [\phi_0]^{-1} \cdot [\phi_1(x_1)]^{-1} \cdots [\phi^{n-1}(x_1, \ldots, x_{n-1})]^{-1}.$$

Our aim is to investigate the joint pdfs of the random vectors $(X_1, \ldots, X_n)$, which are the images of the random vectors $(T_1, \ldots, T_n)$ under the transformations (1). These can easily be obtained using standard methods, and accordingly to the class the distributions all T_i 's belong to, one obtains generalizations of those classes i.e., n-variate ex-normal , ex-exponential, ex-Weibullian or ex-gamma pdfs respectively.

The ex-normals (under the name "pseudonormals") were explored in [Filus and Filus, 2000], [Filus and Filus, 2001b], [Filus and Filus, 2001a] (see also [Kotz *et al.*, 2000], pages 217-218). The other classes will be investigated in this paper in association with system reliability and maintenance modeling.

2 The n-variate three parameter ex-Weibullian probability densities

Suppose the transformations (1) are applied to the random vectors $(T_1, \ldots, T_n)$ whose independent marginals are distributed according to, in general distinct, three parameter Weibull pdfs $f_1(t_1), \ldots, f_n(t_n)$ respectively. Thus, for $i = 1, \ldots, n$ we have:

$$f_i(t_i) = \begin{cases} (\gamma_i/\beta_i)(t_i - \alpha_i)^{\gamma(i)-1} \exp\left[-(t_i - \alpha_i)^{\gamma(i)}/\beta_i\right], & \text{for } t_i > \alpha_i, \\ 0, & \text{elsewhere,} \end{cases} \tag{2}$$

where the convention $\gamma(i) = \gamma_i$ is to be adopted. The densities (2) will also be denoted by $W(\alpha_i; \beta_i, \gamma(i))$. Using standard procedures one easily obtains the pattern, for ex-Weibullian pdfs of the random vectors $(X_1, \ldots, X_n)$ present in the formula (1), in the following factored form:

$$g(x_1, \ldots, x_n) = g_1(x_1) \cdot g_2(x_2|x_1) \cdots g_n(x_n|x_1, \ldots, x_{n-1}). \tag{3}$$

As it turns out, all the n factors are Weibullian pdfs. So the (initial) pdf of X_1 is

$$g_1(x_1) = W(\phi_0\alpha_1 + \theta_0; \beta_1(\phi_0)^{\gamma(1)}, \gamma_(1)), \tag{4}$$

while for each $j = 2, \ldots, n$, the conditional pdf $g_j(x_j|x_1, \ldots, x_{j-1})$ present in (3) is also Weibullian with respect to x_j alone i.e.,

$$\begin{aligned} g_j(x_j|x_1, \ldots, x_j) = W\,\big(\phi_{j-1}(x_1, \ldots, x_{j-1}) \cdot \alpha_j + \theta_{j-1}(x_1, \ldots, x_{j-1})\,; \\ \beta_j \cdot [\phi_{j-1}(x_1, \ldots, x_{j-1})]^{\gamma(j)}, \gamma(j)\big) \end{aligned} \tag{5}$$

or, more concisely, as well as more generally (see further the "method of parameter replacement"), as:

$$g_j(x_j|x_1, \ldots, x_{j-1}) = W(A_j(x_1, \ldots, x_{j-1}); B_j(x_1, \ldots, x_{j-1}), \gamma(j)). \tag{6}$$

In practical situations the values $x_1, \ldots, x_{j-1}$ may often be considered 'fixed' (at the "time instant" j).

3 Pseudogamma probabitity densities

The pattern (1), when applied to the random vectors of independent random variables $(T_1, \ldots, T_n)$ distributed as three parameter gammas, produces other interesting class of joint probability distributions of the random vectors $(X_1, \ldots, X_n)$. As before, denote the pdfs of the n random variables T_i by $f_i(t_i)$ for $i = 1, \ldots, n$. This time we have:

$$f_i(t_i) = \begin{cases} \left[\Gamma(\gamma_i) \cdot (\beta_i)^{\delta_i}\right]^{-1} (t_i - \alpha_i)^{\delta(i)-1} \exp\left[-(t_i - \alpha_i)/\beta_i\right], & \text{for } t_i \geq \alpha_i, \\ 0, & \text{elsewhere,} \end{cases} \tag{7}$$

where the constants α_i are the shift parameters, and the positive reals β_i and $\delta(i)$ are the scale and the shape parameters respectively. Denote the pdfs $f_i(t_i)$ in (7) by $G(\alpha_i; \beta_i, \delta(i))$. The method of the construction of the joint pdf of any random vector $(X_1, \ldots, X_n)$ defined by (1) is exactly the same as that for the ex-Weibullians. The general formula for the joint pdf $g(x_1, \ldots, x_n)$ of $(X_1, \ldots, X_n)$ has also the factored form (3). Now $g_1(x_1)$ is the gamma pdf:

$$G(\theta_0 + |\phi_0|\alpha_1; |\phi_0|\beta_1; \delta(1)), \tag{8}$$

while for $j = 2, \ldots, n$, the conditional pdfs in (3) are:

$$\begin{aligned} g_j(x_j|x_1, \ldots, x_{j-1}) = G\,(&\theta_{j-1}(x_1, \ldots, x_{j-1}) + |\phi_{j-1}(x_1, \ldots, x_{j-1})| \cdot \alpha_j; \\ &|\phi_{j-1}(x_1, \ldots, x_{j-1})| \cdot \beta_j; \delta(j))\,, \end{aligned} \tag{9}$$

or in a more general form:

$$g_j(x_j|x_1, \ldots, x_{j-1}) = G(A_j(x_1, \ldots, x_{j-1}); B_j(x_1, \ldots, x_{j-1}); \delta(j)). \tag{10}$$

They are the ordinary three parameter gamma densities each considered as a function of the argument x_j only. For this reason the so obtained n-variate pdfs are proposed to be called ex-gamma.

4 Comments

A. Notice that in both the new pdf classes construction, described above, the vectors of shape parameters $(\gamma(1), \ldots, \gamma(n))$, $(\delta(1), \ldots, \delta(n))$ in ex-Weibullian and ex-gamma cases respectively are invariant with respect to the pseudoaffines (1). Therefore their values may stand as a criterion for classification of the ex-Weibullians or ex-gammas. In particular, the vector shape parameters $(1, \ldots, 1)$ uniquely determines the class of the two or one parameter ex-exponentials (the set theoretical intersection of ex-Weibullians and ex-gammas classes), while the vector $(2, \ldots, 2)$ determines subclass of ex-Rayleigh among the ex-Weibullians.

B. Occasionally, it is worth to mention an interesting theoretical fact that for any random vector $(T_1, \ldots, T_n)$ having ex-Weibullian or ex-gamma pdf its image $(X_1, \ldots, X_n)$ under (1) is also ex-Weibullian or ex-gamma respectively (see [Filus and Filus, 2003b] for more details).

C. Each of the considered above n-variate three parameters ex-Weibullian, as well as, each of the ex-gamma pdfs are uniquely determined by one of the two sets of the formulas i.e., by (1), (3), (4), (5) together with (2), or by (1), (3), (8), (9) with (7) respectively. The method described above will be called “transformation method”. The use of the pseudoaffine transformations is mathematically an elegant way to define ex-Weibullians or ex-gammas. There is also another way to obtain the same pdfs, namely when the formula (1) in the above two lists is dropped. Moreover, significantly wider classes of ex-Weibullians and ex-gammas, that properly contain the corresponding classes defined by the transformation method may be obtained. This will happen when one replaces the defining formulas (5) and (9) by more general (6) and (10). Actually, in this case, both the classes of the pdfs are uniquely determined by a choice of the corresponding classes of functions $A_j(x_1, \ldots, x_{j-1})$, $B_j(x_1, \ldots, x_{j-1})$. The considered classes of the pdfs may even be more extended if, in (6) and (10) respectively, also the set of (constant) shape parameters $\gamma(j)$ is enlarged by properly chosen set of "shape parameter functions" $C_j(x_1, \ldots, x_{j-1})$. Therefore two distinct methods of the construction are available. The second method that relies on a proper conditioning, we propose to call "method of parameters replacement". The type of conditioning we apply somehow corresponds to the conditioning pattern used, for example, in [Arnold *et al.*, 1992], as well as in [Arnold and Strauss, 1988], [Arnold and Strauss, 1991] and in many other related papers (see references in the first cited position). On the other hand, those ideas essentially differ from the ones, described in our work. In the setting, outlined above, the following two rules make our conditioning method distinct from these presented in the above references: a) the predetermined order in conditioning (see formula (3)), with exactly $n-1$ conditional pdfs chosen to be specified, is imposed b) these $n-1$ conditional pdfs are always completed by exactly one (initial) marginal pdf ($g_1(x_1)$ in (3)). This is noteworthy that, using the method of parameter replacement, the resulting n-variate pdfs are uniquely characterized and constructed in a very simple way by (3), (4) and (6) in the Weibullian case, and by (3), (8) and (10) in the gamma case respectively. Briefly speaking, this second method of construction allows, in a largely "arbitrary" but unique way, to achieve the modeling goals simply by replacing some constant parameters in pdfs, say, $f_j(t_j)$ of the, already considered, independent random variables T_j, by properly chosen continuous functions of the arguments, say, $x_1, \ldots, x_{j-1}$, while 'formally' replacing t_j by x_j.

5 On reliability applications

Constructions of the new pdfs, carried out in this work, have their origin (see [Filus and Filus, 2003b]) in the set of problems associated with stochastic modeling of reliability of multicomponent parallel systems with stochastically dependent life times $X_1, \ldots, X_n$ of the components (for reliability references see for example [Barlow and Proschan, 1975]). As models for such systems the joint probability distributions of the component life times are frequently applied (see, for example [Freund, 1961], [Marshall and Olkin, 1967], [Lu, 1989], and others; see also [Filus, 1991]; for much more exhaustive references see [Kotz *et al.*, 2000]). Even as in the past more then four decades, numerous models in the form of multivariate probability distributions have been invented, various types of old and new physical or biological systems still require models of that type. The two classes of multivariate pdfs here presented are (to our best knowledge) new as both: the mathematical entities, and as a way of stochastic description of physical dependencies between the components. Roughly speaking, in the models of stochastic dependencies, presented, an assumed mechanism of system behavior relies on the following: if one (or more) of the system components, say, e_i fails then some survived component (or set of components) e_j ($j \neq i;\ i, j = 1, \ldots, n$) keeps a memory of the (random) time X_i of their mutual cooperation that affected conditional pdf $g_j(x_j|x_i)$ (one of those given by (5), (6) or (9), (10)) of its life time X_j, given $X_i = x_i < X_j$. It is assumed that the component e_i by its activity changes the environment or work conditions of the component e_j. The continuous influence of e_i on e_j causes either an improvement or a deterioration in the components e_j functioning, so that these changes, during the time $X_i = x_i$, cause the life time X_j of e_j to become statistically longer or shorter than its "original" life times, say, T_j, under "laboratory conditions" (i.e., in an absence of any other component influences). The underlying Weibullian and gamma conditional pdfs were already discussed in this text. Notice also that the laboratory condition life times $T_1, \ldots, T_n$ may be considered to be independent Weibullian or gamma as those described in Section 1. Here, physical act of installation of the set of separate components into a real system may be thought off as, in a way, corresponding to the mathematical relationship (1) between the random vectors $(T_1, \ldots, T_n)$ and $(X_1, \ldots, X_n)$. For a more exhaustive description of such systems together with a stochastic reasoning, on how to model them, see [Filus and Filus, 2003b].

6 On ex-Weibullian stochastic processes

The dimension n of the space $\mathbf{R}^n$ associated with the pattern of the pseudoaffine transformations (1) may be extended, in a natural way, to infinity (i.e., by letting $n \to \infty$). In such a case the infinite version of (1) may be specified as follows:

$$\begin{aligned}
X_1 &\stackrel{d}{=} \phi_0 T_1 + \psi_0, \\
&\ldots\ldots\ldots \\
X_j &\stackrel{d}{=} \phi_{j-1}(X_1, \ldots, X_{j-1}) T_j + \psi_{j-1}(X_1, \ldots, X_{j-1}), \qquad (11) \\
&\ldots\ldots\ldots
\end{aligned}$$

where $j = 2, 3, \ldots$

Using (11) one obtains new classes of stochastic processes $\{X_1, X_2, \ldots\}$ corresponding to some well known processes $\{T_1, T_2, \ldots\}$ chosen. When assuming that all the random variables $T_1, T_2, \ldots$ are independent Weibullian a class of ex-Weibullian random processes $\{X_j\}$, with discrete time $j = 1, 2, \ldots$ is obtained. Notice that, also in a more general case, if all the parameter functions $\phi_{j-1}(\cdot)$, $\psi_{j-1}(\cdot)$ depend on X_{j-1} only, while all the input random variables $T_1, T_2, \ldots$ are independent, the obtained stochastic process (including the ex-Weibullian) will be Markovian (see Proposition 1 in [Filus and

Filus, 2003a]). For such a Markovian case new extensions of the (discrete and continuous time) normal, in particular extensions of the Wiener stochastic processes, are presented in [Filus and Filus, 2003a]. On the other hand, a variety of non-Markovian cases are available too. In the next, an application of the above ex-Weibullian model to some maintenance problems associated with repairable systems will be presented. For this purpose the stochastic processes, chosen as models, will be deliberately assumed to be highly non-Markovian in the sense that all the parameter functions in the defining formula (11) will essentially depend on all the 'previous' random variables $X_1, \ldots, X_{j-1}$.

7 The maintenance models

Suppose, that after each failure, a system is repaired with a possibility of a choice among a finite number of kinds of repair available. These repairs differ each other by a quality of the repair on one side and by costs on the other. For simplicity, the state of the system at any time is assumed to be known. Also, the time- length of repairs are not included in this simplified setting. Let the stochastically dependent times of system functioning between $(j-1)$-th and j-th failure be modeled by X_j, $j = 1, 2, \ldots$ One of the basic features of the emerging new methodology is the following. Suppose that, for some j, a $(j-1)$-th failure occurred. Also suppose, all the "maintenance history" of the system performance i.e., the times $X_1, X_2, \ldots, X_{j-1}$ of work between the previous failures, and the corresponding sequence of kinds of repair $r_1, r_2, \ldots, r_{j-2}$ applied, is recorded. One of the main questions, that may arise at this point, can be stated as follows: what would be the pdf (or just an expectation) of the time X_j 'from now' to the next failure, if an r_{j-1}-th kind of the repair would be chosen? To get an answer, in the considered framework, one of the Weibull conditional pdfs $g(x_j|x_1, \ldots, x_{j-1})$ of X_j, given by (5) or (6) may be applied as a proposed model. In particular, one may consider the following conditional pdf:

$$\begin{aligned} & g_j(x_j|x_1, \ldots, x_{j-1}) = \qquad (12) \\ & \left[\lambda\left(1 + a_{1,k(1)} x_1^{\beta_1} + a_{2,k(2)} x_2^{\beta} + \cdots + a_{j-1,k(j-1)} x_{j-1}^{\beta}\right)\right] x_j^{\gamma-1} \\ & \exp\left\{-\left[\lambda\left(1 + a_{1,k(1)} x_1^{\beta_1} + a_{2,k(2)} x_2^{\beta} + \cdots + a_{j-1,k(j-1)} x_{j-1}^{\beta}\right)\right] x_j^{\gamma}\right\}, \end{aligned}$$

where all the coefficients present in (12) are positive, and each coefficient $a_{i,k(i)}$ depends on the choice of $r_{k(i)}$-th kind of repair that took place directly after an i-th failure, $i = 1, \ldots, j-1$. If one seeks the best policy for choices of the repairs after the failures a set of optimization problems emerges. In particular, a possible aim, that may be considered, would be to balance system efficiency (in sense of maximizing length of the times $X_1, X_2, \ldots$) against total cost of the repairs, in order to attain a maximal expected profit from the systems exploitation. Other model, an alternative to (12), can also be considered using the following class of the conditional (Weibullian in x_j) pdfs:

$$g_j(x_j|x_1,\ldots,x_{j-1}) = \tag{13}$$
$$\left[\lambda \exp\left(b_{1,k(1)}x_1^{\beta_1} + b_{2,k(2)}x_2^{\beta} + \cdots + b_{j-1,k(j-1)}x_{j-1}^{\beta}\right)\right] x_j^{\gamma-1}$$
$$\exp\left[-\lambda \exp\left[\left(b_{1,k(1)}x_1^{\beta_1} + b_{2,k(2)}x_2^{\beta} + \cdots + b_{j-1,k(j-1)}x_{j-1}^{\beta}\right)\right] x_j^{\gamma}\right],$$

where the coefficients $b_{i,k(i)}$, $(i = 1,\ldots,j-1)$ are arbitrary (possibly also negative). Somewhat simplified versions of the models (12), and (13) one obtains if only the conditional expectations of the life times are of interest. Then, for $j = 2, 3, \ldots$ we have the regressions

$$E[X_j|x_1,\ldots,x_{j-1}] = \tag{14}$$
$$\left[\lambda\left(1 + a_{1,k(1)}x_1^{\beta_1} + a_{2,k(2)}x_2^{\beta} + \cdots + a_{j-1,k(j-1)}x_{j-1}^{\beta}\right)\right]^{-1/\gamma} \Gamma(1+1/\gamma),$$

and

$$E[X_j|x_1,\ldots,x_{j-1}] = \tag{15}$$
$$\left\{\lambda \exp\left[b_{1,k(1)}x_1^{\beta_1} + b_{2,k(2)}x_2^{\beta} + \cdots + b_{j-1,k(j-1)}x_{j-1}^{\beta}\right]\right\}^{-1/\gamma} \Gamma(1+1/\gamma).$$

Obviously the expectations (14), (15) correspond to the pdfs (12), (13) respectively. Both cases simplify to the exponential cases when $\gamma = 1$.

8 Analytic examples

Because of the space limitation we only mention that numerous nice examples of the new bivariate pdfs with easy analytical calculations can be given. For more on that we refer readers to [Filus and Filus, 2003b].

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