Centered Semi-Markov Random Walk in Diffusion Approximation Scheme

Vladimir S. Koroliuk[1] and Nikolaos Limnios[2]

[1] Institute of Mathematics,
National Academy of Sciences,
Kiev 01001, Ukraine
[2] Laboratoire de Mathématiques Appliquées,
Université de Technologie de Compiègne,
B.P. 20529, 60205 Compiègne Cedex, France

Abstract. In this paper we present a diffusion approximation algorithm for a centered semi-Markov random walks in the series scheme with the small parameter series $\varepsilon \to 0$, $(\varepsilon > 0)$.

1 The semi-Markov random walk

The semi-Markov random walk (SMRW) is defined on the real line $\mathbf{R} = (-\infty, +\infty)$ by the superposition of two independent renewal processes of the i.i.d. sequences of nonnegative random variables $\alpha_k^{\pm}$, $k \geq 1$, and the two sequences of nonnegative independent i.i.d. random variables $\beta_k^{\pm}, k \geq 1$, as follows

$$\zeta(t) = u + \sum_{k=1}^{\nu^+(t)} \beta_k^+ - \sum_{k=1}^{\nu^-(t)} \beta_k^-, \quad t \geq 0. \tag{1}$$

The renewal process are

$$\nu^{\pm}(t) := \max \left\{ n : \sum_{k=1}^{n} \alpha_k^{\pm} \leq t \right\}, \quad t \geq 0. \tag{2}$$

The distribution functions

$$P_{\pm}(t) = P\{\alpha_k^{\pm} \leq t\}, \quad G_{\pm}(u) = P\{\beta_k^{\pm} \leq u\} \tag{3}$$

are given.

SMRW (1) was investigated in average, diffusion and Poisson approximation schemes under distinct assumption of semi-continuity [Korolyuk and Korolyuk, 1999], [Korolyuk, 1997], [Korolyuk, 1999], etc. This kind of processes are interesting for various applied problems. The number of customs in the queue system is described by (1) with the given distribution function of arrival and service time and with . The process (1) can be considered as a mathematical model of risk process with arbitrary distribution of interval between moments of payment of claims and the premium income.

In this paper we discuss a centered normalized in diffusion approximation scheme (see process (12) below).

2 The superposition of two renewal processes

Relation (2) can be described by the counting process

$$\nu(t) = \nu^+(t) + \nu^-(t), \quad t \geq 0, \tag{4}$$

for the Markov renewal process $x_n, \tau_n, n \geq 0$, on the phase space $E = E_+ \cup E_-, E_\pm = \{\pm, x > 0\}$ by the formula of sojourn times $\theta_{n+1} := \tau_{n+1} - \tau_n, \quad n \geq 0$ [Korolyuk and Korolyuk, 1999]:

$$\theta_x^\pm = \alpha^\pm \wedge x. \tag{5}$$

The transition probabilities of the **embedded Markov chain (EMC)** $x_n, \theta_n, n \geq 0$, is defined by the matrix [Korolyuk and Limnios, (2004b]

$$P(x, dy) = \begin{pmatrix} P_+(x - dy) & P_+(x + dy) \\ P_-(x + dy) & P_-(x - dy) \end{pmatrix}.$$

The stationary distribution of the EMC has the density

$$\rho_\pm(t) = \overline{P}_\mp(t)/a, \quad a := a_+ + a_-, \quad a_\pm := \mathbf{E}\alpha^\pm. \tag{7}$$

As usual, $\overline{P}_\pm(t) := 1 - P_\pm(t)$.

The embedded SMRW $\zeta_n := \zeta(\tau_n), n \geq 0$, is defined by the relations

$$\begin{gathered} \zeta_{n+1} = \zeta_n + \beta_{n+1}, \quad n \geq 0, \\ \beta_{n+1} := \beta_{n+1}^+ I(x_{n+1} \in E_+) - \beta_{n+1}^- I(x_{n+1} \in E_-), \end{gathered} \tag{8}$$

where, as usual, $I(A)$ is the indicator of a random event A.

The SMRW (1) can be defined as follows: $\zeta(t) = \zeta_{\nu(t)}, t \geq 0$. It is worth noticing that the average drift per unit time of the SMRW (1) is defined by the value

$$b = b_+/a_+ - b_-/a_-, \quad b_\pm := E\beta^\pm. \tag{9}$$

The average algorithm for SMRW (1) is realized in the following series scheme with the small series parameter $\varepsilon \to 0$ ($\varepsilon > 0$):

$$\zeta_\varepsilon(t) = u + \varepsilon \sum_{k=1}^{\nu^+(t/\varepsilon)} \beta_k^+ - \varepsilon \sum_{k=1}^{\nu^-(t/\varepsilon)} \beta_k^-, \quad t \geq 0. \tag{10}$$

Under the condition $b \neq 0$, the weak convergence takes place:

$$\zeta_\varepsilon(t) \Rightarrow \zeta_0(t) = u + bt, \quad \varepsilon \to 0. \tag{11}$$

3 The algorithm of diffusion approximation

The centered SMRW in the series scheme is considered as follows:

$$\zeta^\varepsilon(t) = u + \varepsilon \left[\sum_{k=1}^{\nu^+(t/\varepsilon^2)} \beta_k^+ - \varepsilon \sum_{k=1}^{\nu^-(t/\varepsilon^2)} \beta_k^- \right]^+ - b\tau(t/\varepsilon^2). \tag{12}$$

The renewal process $\tau(t) := \tau_{\nu(t)}, t \geq 0$, defines the last renewal moment before time t.

Introduce the random variables

$$\gamma_n := \beta_n - b\theta_n, \quad n \geq 1, \tag{13}$$

it is worth noticing that, for any $x \in E_\pm$,

$$b_\pm(x) := \mathbf{E}[\beta_{n+1}|x_n = x] = \pm[b_\pm P_\pm(x) - b_\mp \overline{P}_\pm(x)] \tag{14}$$

and

$$\widetilde{b}_\pm(x) := \mathbf{E}[\gamma_{n+1}|x_n = x] = b_\pm(x) - ba_\pm(x), \tag{15}$$

where

$$a_\pm(x) := \mathbf{E}[\theta_{n+1}|x_n = x] = \int_0^\infty \overline{P}_\pm(t)dt. \tag{16}$$

The centered SMRW (12) can be represented in the following form:

$$\zeta^\varepsilon(t) = u + \varepsilon \sum_{n=1}^{\nu(t/\varepsilon^2)} \gamma_n, \quad t \geq 0. \tag{17}$$

Theorem 1 *Let $b \neq 0$ defined in (9) and the third moments $E[\beta_n^\pm]^3 < \infty$. Then the weak convergence*

$$\zeta^\varepsilon(t) \Rightarrow \zeta^0(t) = u + \sigma w(t), \quad \varepsilon \to 0 \tag{18}$$

takes place. The variance σ^2 of the standard Wiener process $w(t)$ is calculated by the formulae:

$$\begin{array}{c} \sigma^2 = \sigma_0^2 + \sigma_1^2 - \sigma_2^2, \\ \sigma_0^2 = q\int_0^\infty [\rho_+(x)C_+(x) + \rho_-(x)C_-(x)]dx, \\ \sigma_1^2 = 2\int_0^\infty [\pi_+(x)h_+(x) + \pi_-(x)h_-(x)]dx, \\ \sigma_2^2 = 2q\int_0^\infty [\rho_+(x)\widetilde{b}_+^2(x) + \rho_-(x)\widetilde{b}_-^2(x)]dx. \end{array} \tag{19}$$

Here, by definition:

$$C_\pm := E[\gamma_{n+1}^2|x_n = x],$$

$$h_\pm := -\widetilde{b}_\pm^0(x)R_0^\pm\widetilde{b}_\pm^0(x), \widetilde{b}_\pm^0(x) := \widetilde{b}_\pm/a_\pm(x),$$

$$\pi_\pm(x) := q\rho_\pm(x)a_\pm(x), q := 1/a_+ + 1/a_-,$$

where $x \in E_\pm$.

The potential operator $R_0^\pm$ is defined for the generator of the Markov kernel

$$Q = q(x)[P - I].$$

Remark. It is worth noticing that $\sigma_1^2 - \sigma_2^2 \geq 0$.

4 Scheme of Proof

The construction of the algorithm of diffusion approximation is realized by the scheme introduced in our papers [Korolyuk and Limnios, 2004a] and [Korolyuk and Limnios, (2004b].

The compensating operator $\mathbf{L}^\varepsilon$ of the extended Markov renewal process

$$\zeta_n^\varepsilon := \zeta^\varepsilon(\tau_n^\varepsilon), \quad x_n, \quad \tau_n^\varepsilon := \varepsilon^2 \tau_n, \quad n \geq 0 \tag{20}$$

on the test-function $\varphi(u, \cdot) \in C^3(R)$ admit the asymptotic representation

$$\mathbf{L}^\varepsilon \varphi(u,x) = [\varepsilon^{-2} Q + \varepsilon^{-1} Q_1(x) + Q_2(x)]\varphi(u,x) + \theta_l^\varepsilon \varphi(u,x) \tag{21}$$

where

$$Q_1(x)\varphi(u) = q(x)P\widetilde{b}(x)\varphi'(u), \tag{22}$$

$$Q_2(x)\varphi(u) = \frac{1}{2}q(x)PC(x)\varphi''(u), \tag{23}$$

and the remainder operator θ_l^ε satisfies the negligible condition:

$$||\theta_l^\varepsilon \varphi(u)|| \to 0, \varepsilon \to 0, \varphi(u) \in C^3(R). \tag{24}$$

The limit operator

$$\mathbf{L}\varphi(u) = \frac{1}{2}\sigma^2 \varphi''(u)$$

is determined by a solution of the singular perturbation problem for the truncated operator

$$\mathbf{L}_0^\varepsilon \varphi^\varepsilon := [\varepsilon^{-2}Q + \varepsilon^{-1}Q_1 + Q_2](\varphi(u) + \varepsilon\varphi_1(u,x) + \varepsilon^2\varphi_2(u,x)) = \mathbf{L}\varphi(u) + \theta_0^\varepsilon \varphi(u). \tag{25}$$

According to Lemma 3.3 [Korolyuk and Korolyuk, 1999] (p.51) the operator $\mathbf{L}$ in (25) is calculated by the formula

$$\mathbf{L}\Pi = \Pi Q_2 \Pi - \Pi Q_1 R_0 Q_1 \Pi, \tag{26}$$

where the projector Π is defined by the stationary distribution of the associated Markov process with the generator $Q = q(x)[P - I], q(x) = 1/m(x), m(x) := E\theta_x$.

After some computation we obtain the result of Theorem 1.

The verification of the algorithm of diffusion approximation follows some familiar procedure in the theory of convergence of stochastic processes [Ethier and Kurtz, 1986], [Jacod and Shiryaev, 1987], adapted to the semi-Markov switching process in [Korolyuk and Limnios, 2002a], [Korolyuk and Limnios, 2004a], [Korolyuk and Limnios, (2004b].

References

[Bratiichuk, 1995]N.S. Bratiichuk (1995). Limit theorems for some characteristics of system In *Exploring Stochastic Laws*, VSP, 77-90.

[Ethier and Kurtz, 1986]S.N. Ethier and T.G. Kurtz (1986). *Markov Processes: Characterization and convergence*, J. Wiley, New York.

[Jacod and Shiryaev, 1987]J. Jacod and A.N. Shiryaev (1987). *Limit Theorems for Stochastic Processes*, Springer-Verlang, Berlin.

[Korolyuk, 1997]V.S. Korolyuk (1997). Ruin Problems. Explicit and asymptotic approaches. *Theory of Stoch. Processes*, 2 (18), No.1-2, 4-14.

[Korolyuk, 1999]V. S. Korolyuk (1999). Semi-Markov random walk, In Semi-Markov Models and Applications, J. Janssen, N. Limnios (Eds.), pp 61– 75, Kluwer, Dordrecht.

[Korolyuk and Korolyuk, 1999]V. S. Korolyuk and V. V. Korolyuk (1999).

[Korolyuk and Limnios, 2002a]V. S. Korolyuk and N. Limnios, (2002). “Markov additive processes in a phase merging scheme”, *Theory Stochastic Processes*, vol. 8, no 24, pp 213–226.

[Korolyuk and Limnios, 2002b]V. S. Korolyuk, N. Limnios, (2002). “Poisson Approximation of Homogeneous Stochastic Additive Functionals with Semi-Markov Switching”, *Theory of Probability and Mathematical Statistics*, vol. 64, pp 75–84.

[Korolyuk and Limnios, 2004a]V. S. Korolyuk and N. Limnios, (2004). “Poisson approximation of increment processes with Markov switching”, *Theor. Probab. Appl.*, 49(4), 1-18.

[Korolyuk and Limnios, 2004c]V. S. Korolyuk and N. Limnios, (2004). “Average and diffusion approximation for evolutionary systems in an asymptotic split phase space”, *Annals Appl. Probab.*, 14(1), pp 489–516.

[Korolyuk and Limnios, (2004b]V. S. Korolyuk, N. Limnios, (2004). ”Semi-Markov random walk in Poisson approximation scheme”, *Communication in Statistics - Theory and Methods*, 33(3), pp 507–516.

[Prabhu, 1980]N.U. Prabhu (1980). *Stochastic Storage Processes*, Springer-Verlag, Berlin.

[Silvestrov, 2004]D. S. Silvestrov (2004). *Limit Theorems for Randomly Stopped Stochastic Processes.* Series: Probability and its Applications, Springer.